Relations.

Definition 1. An *n-ary relation* (or *relation of arity* n is a subset of a Cartesian product of n factors.

(1) $R \subseteq A_1 \times A_2 \times \cdots \times A_n$ (n factors, $n \geq 1$): R is a relation of arity n **among** (or **between**) $A_1, \ldots, A_n$.
(2) $R \subseteq A \times A \times \cdots \times A$ (n equal factors, $n \geq 1$): R is a relation of arity n **on** A.
(3) $R \subseteq A \times B$ (2 factors): R is a binary relation **from** A **to** B.

If $R \subseteq A \times B$ is a binary relation from A to B, then we may use infix notation, $a\ R\ b$, to mean $(a, b) \in R$. We may write $a_1\ R\ a_2\ R\ \cdots\ R\ a_m$ to mean $(a_1, a_2), \ldots, (a_{n-1}, a_n) \in R$.

Examples.

(1) (Subsets.) A subset of A, $R \subseteq A$, is a unary (= 1-ary) relation on A. For example, of $P \subseteq \mathbb{R}$ is the set of positive real numbers, then P is a unary relation on $\mathbb{R}$.
(2) (Functions.) A *function* from A to B is a binary (= 2-ary) relation $F \subseteq A \times B$ that satisfies the Function Rule, which is:
 for every $a \in A$ there exists exactly one $b \in B$ such that $(a, b) \in F$.
 We often write $F(a) = b$ to mean $(a, b) \in F$.
(3) (Operations.) A *k-ary operation* on A is a function $F : A^k \to A$. As a relation, $F \subseteq A^k \times A = A^{k+1}$ is $(k + 1)$-ary.
 (a) (Addition) $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N} : (m, n) \mapsto m + n$
 is a binary operation on $\mathbb{N}$. As an operation it is binary, but as a relation it is ternary, and it consists of the ordered triples $(m, n, m + n) \in \mathbb{N}^3$.
 (b) (Union) $\cup : \mathcal{P}(A) \times \mathcal{P}(A) \to \mathcal{P}(A) : (X, Y) \mapsto X \cup Y$
 is a binary operation on $\mathcal{P}(A)$.
(4) (Adjacency relation.) Let V be any set (called the set of "vertices") and let $E \subseteq V \times V$ be any binary relation. For each $(u, v) \in E$, draw a directed arrow from u to v. You have now drawn a directed graph. The set E is the "edge relation" or "adjacency relation" of the graph. E is a binary relation on V.
(5) (Incidence relation.) Let $\mathcal{P}$ be the set of points in some geometry and let $\mathcal{L}$ be the set of lines in the geometry. Much of the structure of the geometry is determined by the relation of *incidence*, $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$, which records when a point lies on a line. That is, $(p, \ell) \in \mathcal{I}$ means that p is a point that lies on line ℓ.

Relations versus predicates.

Recall that a predicate is a function into the set of truth values: $\{\bot, \top\}$, $\{F, T\}$, or $\{0, 1\}$. There is a correspondence between relations defined on sets and predicates defined on sets, namely a predicate is the linguistic element used to talk and write about relations, and relations are the concrete realizations of predicates.

(1) A relation is the *support* of its corresponding predicate.
(2) A predicate is the *characteristic function* of its corresponding relation.

(1) The binary relation $\subseteq$ is defined on the set $\mathcal{P}(\{a\})$. Write down all pairs in this relation.

$\subseteq$ is the subset of $\mathcal{P}(\{a\}) \times \mathcal{P}(\{a\})$ consisting of all pairs (x, y) such that $x \subseteq y$. This set is $\{(\emptyset, \emptyset), (\emptyset, \{a\}), (\{a\}, \{a\})\}$. The one pair from $\mathcal{P}(\{a\}) \times \mathcal{P}(\{a\})$ that we do not include in the $\subseteq$-relation is $(\{a\}, \emptyset)$, since $\{a\} \not\subseteq \emptyset$.

(2) How many pairs are in the relation $\subseteq$ when it is defined on $\mathcal{P}(\{a, b\})$? Is this relation a function?

There are four elements in $\mathcal{P}(\{a, b\})$, namely $\emptyset, \{a\}, \{b\}, \{a, b\}$. There are 4 pairs in the $\subseteq$-relation on $\mathcal{P}(\{a, b\})$ which have the form $(\emptyset, \_)$; 2 pairs in the $\subseteq$-relation of the form $(\{a\}, \_)$; 2 pairs of the form $(\{b\}, \_)$; 1 pair of the form $(\{a, b\}, \_)$. Altogether this makes 9 pairs:

$$\{(\emptyset, \emptyset), (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{a, b\}), (\{a\}, \{a\}), (\{a\}, \{a, b\}), (\{b\}, \{b\}), (\{b\}, \{a, b\}), (\{a, b\}, \{a, b\})\}.$$

The $\subseteq$-relation is not a function, since it does not satisfy the Function Rule: $(\emptyset, \{a\})$, $(\emptyset, \{b\})$ are both in relation, they have the same first coordinate, but they do not have the same second coordinate.

(3) Although we have not yet talked about the real numbers $\mathbb{R}$ or the binary ordering symbol $<$ that is defined on this set, you may have heard about them before. Write down three pairs in the relation $<$ and three pairs not in the relation $<$.

$(x, y) = (0, 1), (0, 2)$, or (e, π) are in the $<$-relation, since in each case $x < y$. $(x, y) = (0, 0), (\sqrt{2}, -3)$, or $(e, 1)$ are not in the $<$-relation, since in each case $x \not< y$.

(4) Write down all binary relations on the set $\emptyset$. Then write down all binary relations on the set $\{a\}$. Which binary relations on $\{a\}$ are functions?

A binary relation on $\emptyset$ is a subset of $\emptyset \times \emptyset = \emptyset$. The only such subset is $\emptyset$. Therefore, the only binary relation on $\emptyset$ is the empty relation.

On the other hand, the binary relations on $\{a\}$ are the subsets of $\{a\} \times \{a\} = \{(a, a)\}$. This one element set has two subsets, $\emptyset$ and $\{(a, a)\}$, so there are two binary relations on $\{a\}$:

(a) the empty relation, $\emptyset$, and

(b) the "total" binary relation, $\{a\} \times \{a\} = \{(a, a)\}$.

Both of these relations satisfy the Function Rule, so they are both functions.

(5) What is the arity of the equality relation on $\{a, b\}$? Write down the tuples in this relation. Is the equality relation a function?

The $=$-relation on $\{a, b\}$ is $\{(a, a), (b, b)\}$. This is a binary relation, which is a function.