Practice with Logic and Counting!

(1) One language for ordered sets has $\leq$ as its only nonlogical symbol. In this language, write a formula $\varphi(x)$ expressing that "x is not the largest element and not the smallest element."

(2) Write a formal sentence that expresses Fermat's Last Theorem in a language for number theory whose nonlogical symbols are $0, +, \cdot, \frown, <$. (Fermat's Last Theorem is the statement that if x, y, z, n are nonzero natural numbers and n is at least 3, then $x \frown n + y \frown n = z \frown n$ does not hold.)

(3) If $(a_n)_{n\in\mathbb{N}} = (a_0, a_1, a_2, \ldots)$ is a sequence of real numbers, then we use the abbreviation "$\lim\limits_{n\to\infty} a_n = L$" to mean

$$(\forall k > 0)(\exists N)(\forall n)\left((n > N) \to \left(|a_n - L| < \frac{1}{k}\right)\right).$$

(a) If $(a_n) = (1, 1, 1, \ldots)$, show that $\lim_{n\to\infty} a_n = 1$.

(b) If $(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$, show that $\lim_{n\to\infty} a_n = 0$.

(c) If $(a_n) = (1, 2, 3, \ldots)$, show that $\lim_{n\to\infty} a_n \neq 0$.

(4)

(a) Show that if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then $\lim_{n\to\infty}(a_n + b_n) = A + B$.

(b) Expand the following into a formal sentence and then draw the formula tree for that sentence.

$$(((\lim_{n\to\infty} a_n = A) \quad \text{and} \quad (\lim_{n\to\infty} b_n = B)) \quad \text{implies} \quad (\lim_{n\to\infty}(a_n + b_n) = A + B)).$$

(5) Rewrite the following sentences so that they are in prenex form, and the bodies of the sentences are in disjunctive normal form.

(a) $(\forall x)(x = x) \rightarrow (\exists x)(x = x)$.

(b) $(\forall x)(\forall y)((x < y) \rightarrow (\exists z)(x < z < y))$.
(Here $x < z < y$ is an abbreviation for $(x < z) \wedge (z < y)$.)

(6) If possible, give an example of a propositional formula (or "truth function") that uses only the variables P and Q and only the logical connectives $\wedge, \vee, \neg, \rightarrow$ and $\leftrightarrow$, which satisfies the following three properties.

(a) The formula is a tautology,

(b) If you replace every instance of P with $\neg P$ in the formula it remains a tautology, but

(c) If you replace every instance of Q with $\neg Q$ in the formula it changes to a contradiction.

(7) Write an informal explanation of what it means for a proof system to have each of the following characteristics.

(a) Soundness.

(b) Completeness.

(c) Decidability.

(8) Explain the difference between truth and provability.

(9) How many 7-digit numbers have all of these properties?

- All digits are distinct,
- the leading digit is not 0,
- no two consecutive digits are even, and no two consecutive digits are odd.

(10) Thirty dots are evenly spaced on the circumference of a circle. How many ways can we choose a subset of these dots if we must pick at least three dots and we are not allowed to choose exactly the set of vertices of a regular polygon?

(11)

(a) How many binary relations on the set $n = \{0, 1, \ldots, n-1\}$ are there?

(b) How many binary relations on n are reflexive?

(c) How many binary relations on n are reflexive and symmetric?

(d) Explain why there are B_n binary relations on n that are reflexive, symmetric, and transitive.

(12) These problems are about seating people at a round table. Two seating arrangements are considered the same if they differ by a rotation. (So, for example, the arrangement $ABCDEF$ is the same as $BCDEFA$.)

(a) How many ways are there to sit 3 couples at a round table?

(b) What if couples must sit together?

(c) What if couples are not allowed to sit together?

(13) Imagine the following two events concerning ordinary 6-sided dice.

(a) You roll a single die twice and the sum of values is 7.

(b) You roll a single die four times and the sum of values is 14.

Is the first event more probable, equally probable, or less probable than the second event?