

Practice with Inclusion/Exclusion, Stirling, and Bell numbers!

- (1) How many positive integers less than 1000 are not divisible by 2, 3, 5 or 7?

I will count the number of positive integers ≤ 1000 that are not divisible by 2, 3, 5 or 7, since the number is the same for " ≤ 1000 " as " < 1000 ", but some of the divisions are simpler for " ≤ 1000 ".

$$1000 - 500 - 333 - 200 - 142 + 166 + 100 + 71 + 66 + 47 + 28 - 33 - 23 - 14 - 9 + 4 = 228.$$

- (2) In a class of 20 students, how many study groups can be formed which include at least one of the three students Archibald, Beryl, or Cornelia? Assume that a study group must involve at least 2 students.

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - \dots \\ &= (2^{19} - 1) + (2^{19} - 1) + (2^{19} - 1) - 2^{18} - 2^{18} - 2^{18} + 2^{17}. \end{aligned}$$

- (3) How many 5-digit numbers fail to contain the sequence 01? How about 00?

First part:

$$10^5 - 4 \cdot 10^3 + 3 \cdot 10 = 96030.$$

Second part:

$$10^5 - 4 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10 - 22 + 1 = 96309.$$

- (4) How many 6-digit numbers have the property that, for every k , the k th digit is different than the $(7 - k)$ th digit?

$$\binom{3}{0}10^6 - \binom{3}{1}10^5 + \binom{3}{2}10^4 - \binom{3}{3}10^3 = 10^3 9^3 = 729000.$$

- (5) A news organization reports that the percentage of voters who would be satisfied with each of three candidates A , B , C for President is 65%, 57%, 58% respectively. Furthermore, 28% would accept A or B , 30% would accept A or C , 27% would accept B or C , and 12% would accept any of the three. Is this fake news?

Yes, it is fake news. One calculates that 107% of voters support at least one candidate.

- (6) If $f : k \rightarrow k$ is a bijection, then i is called a fixed point of f if $f(i) = i$. What percentage of bijections $f : k \rightarrow k$ have no fixed points? (Count the number of bijections with no fixed points, then divide by the total number of of bijections.)
- (7) Explain why $S(n, 2) = 2^{n-1} - 1$ if $n > 0$.
- (8) Explain why $S(n, n - 1) = \binom{n}{2}$.
- (9) Determine how the numbers 2^{n-1} , B_n , $n!$, 2^{n^2} are related to each other as n grows. (Which is larger than which?)

$$2^{n-1} \leq B_n \leq n! \leq 2^{n^2}$$

for all n .

- ($2^{n-1} \leq B_n$): For $n > 0$, use the answer to Problem 7:

$$2^{n-1} = 1 + \underline{(2^{n-1} - 1)} = S(n, 1) + \underline{S(n, 2)} \leq \sum_{k=0}^n S(n, k) = B_n.$$

If $n = 0$, then $2^{0-1} < 1 = B_0$.

- ($B_n \leq n!$): We describe an injection $f : \text{Part}(n) \rightarrow \text{Perm}(n)$ from the set of partitions of n to the set of permutations of n . There are B_n partitions and $n!$ permutations, so this will establish that $B_n \leq n!$
 Suppose we are given a partition P of n , for example if $n = 6$ we might be given $P = \{\{2, 5\}, \{1, 4, 6\}, \{3\}\}$. Order the cells by least element: $\{1, 4, 6\} < \{2, 5\} < \{3\}$. Then order the cell contents in the reverse of the natural order: $641 < 52 < 3$. Concatenate results in this order to obtain $f(P) = 641523$. The result is a reordering, or permutation, of the numbers $1 \dots 6$. It was done carefully to ensure that f is an injection. (Verify this!)
- ($n! \leq 2^{n^2}$): There are $n!$ bijections $f : n \rightarrow n$. There are 2^{n^2} binary relations $R \subseteq n \times n$. Each bijection is a binary relation, so $n! \leq 2^{n^2}$.