The Principle of Inclusion and Exclusion!

Version 1. The principle of inclusion and exlclusion counts the size of a union.

$$|A_1\cup\cdots\cup A_n| = \sum_{1\le i\le n}|A_i| - \sum_{1\le i<j\le n}|A_i\cap A_j| + \sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k| - \cdots + (-1)^{n+1}|A_1\cap\cdots\cap A_n|.$$

Version 2. Let X be a set and let $\mathcal{P}$ be a set of properties the elements of X may have. If $N_=(S)$ is the number of elements of X that have exactly the properties in $S\subseteq\mathcal{P}$ and $N_\ge(S)$ is the number of elements of X that have at least the properties in $S\subseteq\mathcal{P}$, then

$$\begin{aligned} N_\ge(S) &= \textstyle\sum_{S\subseteq T\subseteq\mathcal{P}} N_=(T) \quad \text{and} \\ N_=(S) &= \textstyle\sum_{S\subseteq T\subseteq\mathcal{P}} (-1)^{|T|-|S|} N_\ge(T). \end{aligned}$$

The first formula is trivial; the principle of inclusion and exclusion is the second formula.

Exercises.

(1) (a) What is the number of surjective functions $f: 5\to 3$? (Hints: Let X be the set of <u>all</u> functions from 5 to 3. Let $\mathcal{P}=\{P_0,P_1,P_2\}$ be the set of properties where P_i is the property of $f\in X$ which says $i\notin \mathrm{im}(f)$. Compute $N_=(\emptyset)$.)

(b) What is the number of surjective functions $f: n\to k$?

(2) How many positive integers less than 1000 are not divisible by 2, 3, 5 or 7?

(3) How many positive integers less than 250 are not perfect powers?

(4) How many 5 digit numbers fail to contain the sequence 01? How about 00?

(5) How many 6 digit numbers have the property that, for every k, the kth digit is different than the $(7-k)$th digit?