DISCRETE MATH
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

(1) Fill in the blank with all appropriate choices (if any exist).

(a) Mathematics is founded on __set theory__.

(i) set theory

(b) Naive set theory is __inconsistent__.

(i) free of any contradictions

(ii) inconsistent

(iii) Zermelo's invention

(c) The sentence

$$\forall x\, \forall y\, \exists p\, \forall z\, ((z \in p) \leftrightarrow (z = x \text{ or } z = y))$$

expresses the Axiom of __(leave blank)__.[1]

(i) Extensionality

(ii) Power Set

(iii) Separation

(2) Define

(a) <u>function</u> from A to B.

A function from A to B is a relation from A to B that satisfies the function rule.

(b) the set of <u>natural numbers</u>.

The set of natural numbers is the intersection of all inductive sets.

(c) <u>finite</u>.

A set is finite if it is equipotent with a natural number. (Or, equivalently, if it has a bijection with a natural number.)

[1] The sentence in 1(c) expresses the Axiom of Pairing.

(3) Give an example, if one exists, of each of the following. If no example exists, say why.

(a) A function $f : \mathbb{N} \to \mathbb{N}$ that is injective, but not surjective.

The successor function, $S : \mathbb{N} \to \mathbb{N} : n \mapsto n \cup \{n\}$ is injective but not surjective.

(b) A partition of $\mathbb{N}$ with exactly two cells.

$\{\{0\}, \{n \in \mathbb{N} \mid n \neq 0\}\}$ is a partition of $\mathbb{N}$ into the singleton zero set and the set of nonzero natural numbers.

(c) A function $f : A \to B$ for which $|\mathrm{coim}(f)| \neq |\mathrm{im}(f)|$.

No such function exists. If $f : A \to B$ is any function, then the induced map $\overline{f} : \mathrm{coim}(f) \to \mathrm{im}(f)$ is a bijection, so $|\mathrm{coim}(f)| = |\mathrm{im}(f)|$.

(4) (a) Write down the recursive definition of addition of natural numbers.

$$\begin{aligned} m + 0 &:= m && \text{(IC)} \\ m + S(n) &:= S(m+n) && \text{(RR)} \end{aligned}$$

(b) Prove that $m + (n + k) = (m + n) + k$ for $m, n, k \in \mathbb{N}$.

We prove this by induction on k.
(Base Case: $k = 0$)

$$\begin{aligned} m + (n + 0) &= m + n && (\mathrm{IC}, +) \\ &= (m + n) + 0 && (\mathrm{IC}, +) \end{aligned}$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{aligned} m + (n + S(k)) &= m + S(n + k) && (\mathrm{RR}, +) \\ &= S(m + (n + k)) && (\mathrm{RR}, +) \\ &= S((m + n) + k) && (\mathrm{IH}) \\ &= (m + n) + S(k) && (\mathrm{RR}, +) \end{aligned}$$