

DISCRETE MATH (MATH 2001)

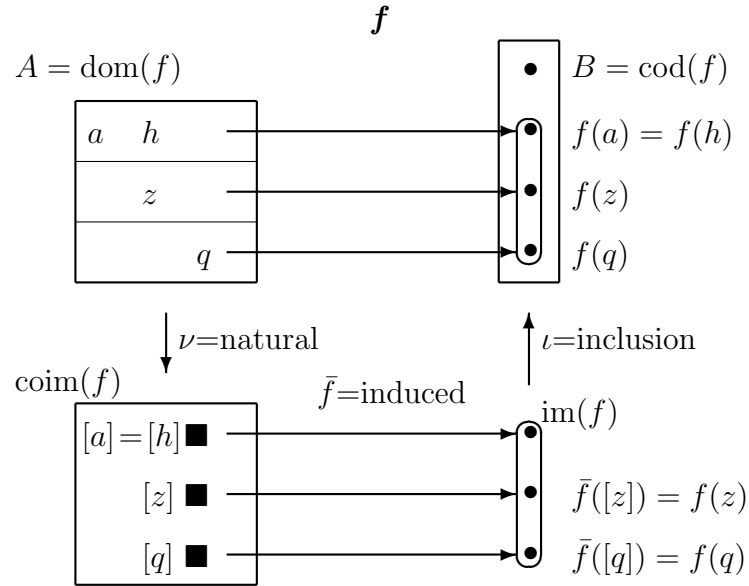
REVIEW SHEET I

I. Set Theory

- (a) Informal notion of a set.
- (b) The axioms of set theory (ZFC).
- (c) Venn diagrams versus the directed graph model of set theory.
- (d) Constructions of new sets (pairing, union, power set, separation, intersection).
- (e) Empty set, successor of a set.
- (f) Inductive sets, natural numbers.
- (g) Naive set theory is inconsistent. Russell's Paradox.
- (h) Classes. The union of a set of sets is a set, while the intersection of a nonempty class of sets is a set.

II. Relations

- (a) Ordered pairs (Kuratowski encoding), triples, and n -tuples. Cartesian product.
- (b) Relations. Directed graph representation of binary relations.
- (c) Connection between relations and predicates.
- (d) Definition of a function. Definition of an operation.



- (e) Domain, codomain, image, coimage. Canonical factorization of a function.
- (f) Inclusion map, identity map, natural map, induced map.
- (g) Injections, surjections, bijections. Composition.
- (h) Coimage versus kernel. Partition versus equivalence relation.

III. Recursion and induction (ordinary and strong).

- (a) \mathbb{N} is the least inductive set.
- (b) Recursion Theorem.
- (c) Induction is a valid form of proof.
- (d) Recursive definitions of arithmetic operations on \mathbb{N} : $x + y, xy, x^y$.
- (e) Use of induction to prove laws of arithmetic.

IV. Cardinality.

- (a) Ordinal numbers versus cardinal numbers.
- (b) Meaning of $|A| \leq |B|$, $|A| = |B|$, and $|A| < |B|$.
- (c) Finite and infinite. Countable and uncountable.
- (d) Cantor-Bernstein-Schroeder Theorem.
- (e) Cantor's Theorem.
- (f) $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = |\mathbb{R}^n|$.

General advice on preparing for a math test.

Be prepared to demonstrate understanding in the following ways.

- (i) Know the definitions of new concepts, and the meanings of the definitions.
- (ii) Know the statements and meanings of the major theorems.
- (iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
- (iv) Know how to perform the different kinds of calculations discussed in class.
- (v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
- (vi) Know how to correct mistakes made on old HW.

Practice Problems.

- (1) How do you answer a question where you are asked to "Give an example"?

Give an example of such a question.

- (2) How do you answer a question where you are asked to "Give a definition"?

Define "definition".

- (3) If you are asked to "Give a proof or counterexample", how do you decide which thing to do?

Give a proof or counterexample to the claim "Every prime is odd."

- (4) Show that if $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- (5) Is it always true that $A \subseteq \mathcal{P}(A)$? If your answer is “No”, is it sometimes true?
- (6) Show that $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.
- (7) What is a function? (Give the definition.)
- (8) For the function $f : \{0, 1, 2\} \rightarrow \{a, b, c\} : 0 \mapsto a, 1 \mapsto a, 2 \mapsto b$, write down each of the following sets.
 - (a) $\text{dom}(f)$
 - (b) $\text{cod}(f)$
 - (c) $\text{im}(f)$
 - (d) $\text{coim}(f)$
 - (e) ν (the natural map, written as a set)
 - (f) \bar{f} (the induced map, written as a set)
 - (g) ι (the inclusion map, written as a set)
 - (h) $\ker(f)$
- (9) Justify the claims that: (i) the squaring function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto x^2$ and the absolute value function $g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto |x|$ have the same kernel and image, but (ii) they are different functions.
- (10) How many functions are there of the form $f : \mathbb{N} \rightarrow \emptyset$? How many functions are there of the form $f : \mathbb{N} \rightarrow \{\emptyset\}$?
- (11) How many different partitions are there on the set $X = \{1, 2, 3\}$? How many different equivalence relations on X are there?
- (12) Give examples of binary relations on \mathbb{N} that are:
 - (a) reflexive and symmetric, but not transitive.
 - (b) reflexive and transitive, but not symmetric.
 - (c) symmetric and transitive, but not reflexive.
- (13) Explain why induction is a valid form of proof.
- (14) Prove that $m(n + k) = (mn) + (mk)$ for all $m, n, k \in \mathbb{N}$.
- (15) Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ by induction.

(16) Show that $\{a\}$ is a finite set. Show that \mathbb{N} is a countable set.

(17) State the theorem.

- (a) Russell's paradox.
- (b) Recursion Theorem.
- (c) Cantor's Theorem.
- (d) Cantor-Bernstein-Schroeder Theorem.

(18) True or False? Explain.

- (a) If $A \times A = B \times B$, then $A = B$.
- (b) If $A \times B = B \times A$, then $A = B$.
- (c) The class of equivalence relations on \mathbb{N} is a set.
- (d) The intersection of the class of all sets is a set.
- (e) If $A \subseteq B \subseteq C$ and $|A| = |C|$, then $|A| = |B|$.