Discrete Math
Quiz 11

Name:______________________________

You have 10 minutes to complete this quiz. If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show your work, and give adequate explanations.

1. How many natural numbers $\leq 1,000,000$ are either perfect squares or perfect cubes? ($x \in \mathbb{N}$ is called a perfect square if $\mathbb{N}$ satisfies $(\exists y)(x = y^2)$, and $x \in \mathbb{N}$ is called a perfect cube if $\mathbb{N}$ satisfies $(\exists z)(x = z^3)$.)

Let S be the set of perfect squares contained in $\{0, 1, \ldots, 10^6\}$ and let C be the set of perfect cubes contained in $\{0, 1, \ldots, 10^6\}$. We want to compute $|S \cup C| = |S| + |C| - |S \cap C|$.

It helps to see that $|S|$ can be counted by counting the set of square roots of elements of S, namely

$$|S| = |\{\lfloor\sqrt{0}\rfloor, \lfloor\sqrt{1}\rfloor, \ldots, \lfloor\sqrt{10^6}\rfloor\}| = |\{0, 1, \ldots, 10^3\}| = 1 + 10^3.$$

Since $|S| = 1 + \sqrt{10^6} = 1 + 10^3$, and similarly $|C| = 1 + \sqrt[3]{10^6} = 1 + 10^2$, and $|S \cap C| = 1 + \sqrt[6]{10^6} = 1 + 10$, the answer is

$$(1 + 10^3) + (1 + 10^2) - (1 + 10) = 10^3 + 10^2 - 10 + 1 = 1091.$$

2. Determine the number $S(5, 3)$. (Make sure your method for determining the number is clear.)

You can use the recursion $S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$, or you try to count the partitions some other way.

(Solution 1: Recursion.)

$$\begin{aligned} S(5,3) &= S(4,2) + 3 \cdot S(4,3) \\ &= (S(3,1) + 2 \cdot S(3,2)) + 3(S(3,2) + 3 \cdot S(3,3)) = S(3,1) + 5 \cdot S(3,2) + 9 \cdot S(3,3) \\ &= 1 + 5 \cdot 3 + 9 \cdot 1 = 25. \end{aligned}$$

(Solution 2: Some other way.)

Say we want to partition $\{a, b, c, d, e\}$ into 3 cells. The partition types must have the form $a/b/cde$ or $a/bc/de$. There are $\binom{5}{3} = 10$ of the first type (since we only have to choose the cell of size 3 to determine the partition). There are $\frac{1}{2}\binom{5}{1,2,2} = 15$ of the second type (since $\binom{5}{1,2,2}$ counts ordered partitions). Altogether this shows that $S(5, 3) = 10 + 15 = 25$.