Solutions to HW 9.

1. This problem involves a deck of 52 distinct playing cards.

(i) In how many ways can a 13-card bridge hand be dealt from the deck?

$(52)_{13}$, which equals $13!\binom{52}{13}$. Either answer is OK.

(ii) How many different 13-card bridge hands are there?

$\binom{52}{13}$

In this problem, there are $\binom{52}{13}$ ways to choose 13 cards from 52, so this is the number of bridge hands. But there are 13! ways to deal a specific set of 13 cards. (Which card is dealt first? which is dealt second? ETC.)

2.

(i) How many paths are there from the point $(0,0)$ of $\mathbb{R}^2$ to the point $(10,15)$ of $\mathbb{R}^2$ if each path consists of a sequence of steps of length 1 moving in the direction of the positive x-axis or the positive y-axis?

Use the letter X to indicate a step of length 1 in the direction of the positive x-axis, and the letter Y to indicate a step of length 1 in the direction of the positive y-axis. A sample path from $(0,0)$ can be expressed as a sequence, say,

$$XXYYYXXYYYYYYXXYXYXYXXYYY.$$

This is an X,Y-string with 10 X's and 15 Y's. The total number of these strings is $\binom{10+15}{10} = \binom{25}{10} = \binom{25}{15} = \frac{25!}{10!15!}$.

(ii) How many paths are there from the point $(0,0,0)$ of $\mathbb{R}^3$ to the point $(10,15,20)$ of $\mathbb{R}^3$ if each path consists of a sequence of steps of length 1 moving in the direction of the positive x-axis, the positive y-axis or the positive z-axis?

A similar argument yields $\binom{45}{10,15,20} = \frac{45!}{10!15!20!}$

3. Let $MC(n,k)$ be the number "n-multichoose-k". Use a combinatorial argument to show that $MC(n,0) + MC(n,1) + \cdots + MC(n,k) = MC(n+1,k)$.

Solution 1. (A combinatorial argument.) Let D be the set of all distributions of k identical balls to $n+1$ distinct boxes with repetition allowed. $|D| = MC(n+1,k) = \left(\binom{n+1}{k}\right) = \binom{n+k}{k}$.

Partition D into sets $D_0, D_1, \ldots, D_k$ where $D_i \subseteq D$ is the number of distributions where i balls are distributed to the first n boxes, while the remaining $k-i$ balls are distributed to the last box. Since we can distribute i identical balls to the first n boxes in $MC(n,i)$ ways, and we have no choice but to put the remaining $k-i$ balls into the last box, we have $|D_i| = MC(n,i)$. Since we have a partition,

$$MC(n+1,k) = |D| = |D_0| + |D_1| + \cdots + |D_k| = MC(n,0) + MC(n,1) + \cdots + MC(n,k).$$

Solution 2. (Not really a combinatorial argument, but OK.)

$$\begin{aligned}
MC(n,0)+MC(n,1)+\cdots+MC(n,k) &= \tbinom{n-1}{0}+\tbinom{n}{1}+\tbinom{n+1}{2}+\cdots+\tbinom{n+k-1}{k} \\
&= [\tbinom{n}{0}+\tbinom{n}{1}]+\tbinom{n+1}{2}+\cdots+\tbinom{n+k-1}{k} \\
&= [\tbinom{n+1}{1}]+\tbinom{n+1}{2}+\cdots+\tbinom{n+k-1}{k} \\
&= [\tbinom{n+1}{1}+\tbinom{n+1}{2}]+\cdots+\tbinom{n+k-1}{k} \\
&\;\;\vdots \\
&= \tbinom{n+k-1}{k-1}+\tbinom{n+k-1}{k} \\
&= \tbinom{n+k}{k} = MC(n+1,k).
\end{aligned}$$