Solutions to HW 10.

1. You have just given birth to octuplets. How many ways can you name your children if you only like the names Billy Bob, Jim Bob and Sue Bob? (Any child is allowed to receive any of these three names.)

Here the answer depends on whether the children are considered to be identical or distinct. Since they are octuplets, one might want to consider them to be identical. But since they are living beings, one might want to consider them to be distinct.

Answer 1. (children are considered to be identical)

We are distributing 8 identical children (= balls) to 3 distinct names (= boxes), with repetitions allowed, and not all names need to get a child. The answer is

$$MC(3,8) = \binom{3+8-1}{8} = \binom{10}{8} = 45.$$

This answer is OK!

Answer 2. (children are considered to be distinct)

We are distributing 8 distinct children (= balls) to 3 distinct names (= boxes), with repetitions allowed, and not all names need to get a child. The answer is

$$3^8 = 6561.$$

This answer is OK!

2.

(i) How many 4-digit numbers have digits in strictly increasing or strictly decreasing order? (You are counting $abcd$ for $a, b, c, d \in \{0, \dots, 9\}$ such that either $a < b < c < d$ or $a > b > c > d$.)

The number of 4 digit numbers with digits decreasing is $\binom{10}{4}$, since once the subset of 4 digits have been chosen from the possible 10 digits there is exactly one way for them to be written in decreasing order. The same argument holds for 4 digit numbers with increasing digits, except we do not allow 0 to be a leading digit in a 4-digit number. Hence we only choose our 4 digits from the 9 possible nonzero digits: $\binom{9}{4}$ ways. Thus, the answer is $\binom{9}{4} + \binom{10}{4} = 336$ ways.

(ii) How many 4-digit numbers have digits $abcd$ satisfy $a \le b \le c \le d$ or $a \ge b \ge c \ge d$?

This can be done like Part (i) with a couple of modifications. Instead of picking a subset of 4 distinct digits we want to pick a multiset of 4 not necessarily distinct digits. E.g., we are allowed to pick the multiset $\{3, 3, 5, 7\}$, which represents the number 3357, whose digits are nondecreasing.

Combining the idea from the first part of the problem with the idea from the previous paragraph, we get that the number of 4 digit numbers with digits in nonincreasing order is $\left(\!\binom{10}{4}\!\right) - 1 = \binom{13}{4} - 1$. (The -1 is included to eliminate 0000 from the count. This is the only sequence of 4 nonincreasing digits that has a leading 0.)

Similarly, the number of 4 digit numbers with digits in nondecreasing order is $\left(\!\binom{9}{4}\!\right) = \binom{12}{4}$.

The full answer to the problem is given by inclusion/exclusion: the number counted in the previous paragraph plus the number counted in the paragraph before minus those double counted: $\binom{12}{4} + (\binom{13}{4} - 1) - 9 = 1200$. (The -9 is to adjust for the fact that $1111, 2222, \ldots, 9999$ were counted twice.)

3. How many 5-card poker hands have cards of every suit?

This is a problem for inclusion and exclusion. Let $P_\spadesuit$ be the property of a 5-card hand that it contains NO spade (that is, no $\spadesuit$). Define $P_\diamondsuit, P_\heartsuit, P_\clubsuit$ similarly, and let $\mathcal{P} = \{P_\spadesuit, P_\diamondsuit, P_\heartsuit, P_\clubsuit\}$ be the set of these four properties.

We want to compute $N_=(\emptyset)$, the number of hands which have exactly none of the properties. We compute it by first computing the $N_\geq(T)$ numbers and applying the inclusion-exclusion formula. The answer is

$$\begin{aligned} N_=(\emptyset) &= \textstyle\sum_{\emptyset\subseteq T\subseteq\mathcal{P}} (-1)^{|T|} N_\geq(T) \\ &= \binom{52}{5} - \binom{4}{1}\binom{39}{5} + \binom{4}{2}\binom{26}{5} + \binom{4}{3}\binom{13}{5} + \binom{4}{4}\binom{0}{5} \\ &= \textstyle\sum_{i=0}^{4} (-1)^i \binom{4}{i}\binom{52-13i}{5} \\ &= 685464. \end{aligned}$$