Definitions about Function (page 4 of the "functions" slides).

(1) $F \subseteq A \times B$, $\quad F\colon A \to B$, $\quad A \xrightarrow{F} B$.

The first notation expresses only that F is a binary relation from A to B. The second and third notation express that F is a function from A to B, so it is a binary realtion from A to B that satisfies the function rule.

(2) F assigns y to x, $\quad y = F(x)$.

This is to remind us that if $F(x) = y$, then F is assigning to x the value y, not the other way around. (F does not assign x to y, rather it assigns y to x.)

(3) $F\colon A \to B\colon x \mapsto$ (value assigned to x). (E.g., $F\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto x^2$)

This is a description of the "mapsto" symbol, $\mapsto$. This is not simply another type of arrow that can be used interchangeably with $\to$. Rather, the notation

$$F\colon \mathbb{R} \to [-1,1]\colon x \mapsto \sin(x)$$

is expressing that F is a function from the domain $\mathbb{R}$ to the codomain $[-1,1]$ which assigns the value $\sin(x)$ to x. The $\mapsto$ symbol is used to indicate the "formula" or "rule" that defines F.

(4) F is injective: (Equivalently, F is 1-1.)

F is injective if

$$F(a) = F(b) \quad \text{implies} \quad a = b.$$

In the contrapositive (hence equivalent) form, this reads

$$a \neq b \quad \text{implies} \quad F(a) \neq F(b).$$

(5) F is surjective: (Equivalently, F is onto.)

F is surjective if $\operatorname{im}(F) = \operatorname{cod}(F)$. If we refer to the directed graph representation of F, it says that each element of the codomain "receives an arrow head". More formally, in symbols,

$$(\forall b)(\exists a)(b = F(a)).$$

Here b is a variable representing values in the codomain of F and a is a variable representing values in the domain of F.

(6) F is bijective: (Equivalently, F is 1-1 and onto.)

bijective = injective + surjective.

(7) F is invertible:

$F: A \to B$ is invertible if there is a function $G: B \to A$ such that $G \circ F = \mathrm{id}_A$ and $F \circ G = \mathrm{id}_B$.

(8) F is constant:

$F: A \to B$ is constant if it "assumes only one value". More precisely, F is constant if $F \subseteq A \times B$ and $F = A \times \{b\}$ for some $b \in B$. IN symbols, we indicate F is constant by writing

$$(\forall x_1)(\forall x_2)(F(x_1) = F(x_2)).$$

(9) F is the identity function on A:

The identity function on A, written id_A, is the function $\mathrm{id}_A: A \to A: x \mapsto x$. As a relation, it is

$$\mathrm{id}_A = \{(a, a) \in A^2 \mid a \in A\}.$$

(10) F is the inclusion map for a subset $A \subseteq B$:

If A is a subset of B, then the inclusion map from A to B is

$$\iota: A \to B: a \mapsto a.$$

As a set, $\iota = \mathrm{id}_A$.

(11) F is the natural map for a partition P on A:

If P is a partition of A, then the natural map from A to P is

$$\nu: A \to P: a \mapsto [a].$$

This is the function that maps $a \in A$ to the cell of P containing a.

(12) $A \xrightarrow{F} B \xrightarrow{G} C$, or $G \circ F \colon A \to C$.

Here we are writing notation for the composition of F and G. The composite function $G \circ F$ is the function $(G \circ F)(a) = G(F(a))$. We read "$G \circ F$" as "G of F" (sometimes just "G circle F"). The composition is defined by the formula

$$G \circ f = \{(a, c) \in A \times C \mid (\exists b \in B)(((a, b) \in F) \wedge ((b, c) \in G))\}.$$

Example. If $F(x) = x^2$ and $G(x) = \sin(x)$, then $G \circ F(x) = G(F(x)) = \sin(x^2)$.