Types

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The type of an element is the set of all things that can be said about that element. The type of a tuple is the set of all things that can be said about it.

If $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{A}^n$ and $\mathbf{x} = (x_1, \ldots, x_n)$, then

$$\mathrm{tp}_n^{\mathbb{A}}(\mathbf{a}) = \mathrm{tp}_{\mathbb{A}}(\mathbf{a}) = \mathrm{tp}(\mathbf{a}) = \{\varphi(\mathbf{x}) \mid \mathbb{A} \models \varphi[\mathbf{a}]\}$$

Example. Let $\mathbb{A}$ be the L-structure $\langle \omega; < \rangle$. Let Σ be the set of $(L_A \cup \{c\})$-sentences equal to the union of $\mathrm{Th}(\mathbb{A}_A)$ and a set of sentences expressing that (i) c is not the smallest element, (ii) c is not the second smallest element, (iii) ETC. Σ is finitely satisfiable, so it has a model $\mathbb{B}$ that properly extends $\mathbb{A}$ containing an element c that is “infinitely large”. $\mathbb{B} \models \mathrm{Th}(\mathbb{A})$, so $\mathbb{B} \equiv \mathbb{A}$, yet the structures can be distinguished by the fact that $\mathbb{B}$ has an element of type $\mathrm{tp}(c)$ and $\mathbb{A}$ does not.

Abstract types

Definition. A **partial n-type** of a theory T is a set $\Sigma(\mathbf{x})$ of formulas in the fixed string of variables $\mathbf{x} = (x_1, \ldots, x_n)$ such that there is a model $\mathbb{A}$ of T and a tuple $\mathbf{a} \in \mathbb{A}^n$ such that $\mathbb{A} \models \varphi[\mathbf{a}]$ for each $\varphi(\mathbf{x}) \in \Sigma(\mathbf{x})$. ($\Sigma(\mathbf{x}) \subseteq \mathrm{tp}_{\mathbb{A}}(\mathbf{a})$.)

A **complete n-type** is a maximal partial n-type. ($\Sigma(\mathbf{x}) = \mathrm{tp}_{\mathbb{A}}(\mathbf{a})$.)

A (p/c) n-type of $\mathbb{A}$ is defined to be a (p/c) n-type of $\mathrm{Th}(\mathbb{A})$.

An n-type is **realized** in $\mathbb{A}$ if it is the type of some n-tuple of $\mathbb{A}$, else **omitted**.

Comments.

1. If you replace $\mathbf{x}$ with a string $\mathbf{c}$ of new constant symbols, then new concepts correspond to old : 'partial type $\Sigma(\mathbf{x})$' corresponds to 'satisfiable $\Sigma(\mathbf{c})$'; 'complete type' corresponds to 'complete theory'; '$\Sigma(\mathbf{x})$ is realized in $\mathbb{A}$ by $\mathbf{a}$' corresponds to '$\mathbb{A}_{\mathbf{a}}$ is a model of $\Sigma(\mathbf{c})$'.
2. "$\mathbb{A}$ realizes $\Sigma(\mathbf{x})$" is the assertion that $\mathbb{A}$ satisfies the $L_{\infty,\omega}$-sentence

$$(\exists \mathbf{x}) \left(\bigwedge_{\varphi(\mathbf{x}) \in \Sigma(\mathbf{x})} \varphi(\mathbf{x}) \right).$$

Spaces of types

Since complete types of L in the variables $\mathbf{x}$ correspond to complete $L \cup \{\mathbf{c}\}$-theories, we can import everything we learned about spaces of complete theories to speak about spaces of complete types. We get a sequence of Stone spaces connected by continuous "projection maps", or "restriction maps":

$$\mathrm{Spec}(L) \leftarrow \mathrm{Spec}(L(x_1)) \leftleftarrows \mathrm{Spec}(L(x_1, x_2)) \Lleftarrow \cdots$$

(The first projection map of $\mathrm{Spec}(L(x_1, x_2))$ to $\mathrm{Spec}(L(x_1))$ takes a complete 2-type $\Sigma(x_1, x_2)$ and restricts it to the subset $\Sigma(x_1, x_2)|_{x_1}$ of those formulas where x_2 does not appear. $\Sigma(x_1, x_2)|_{x_1}$ will be a complete type.)

If T is a theory, then $S_n(T)$ is the closed subset of $\mathrm{Spec}(L(x_1, \ldots, x_n))$ consisting of n-types of T. Again, we have continuous restrictions:

$$S_0(T) \leftarrow S_1(T) \leftleftarrows S_2(T) \Lleftarrow \cdots$$

Recognizing a partial type

Thm. Let $\Sigma(\mathbf{x})$ be a set of L-formulas in $\mathbf{x}$. TFAE:

1. $\Sigma(\mathbf{x})$ is a partial type of T.
2. $T \cup \Sigma(\mathbf{c})$ is a satisfiable set of $(L \cup \{\mathbf{c}\})$-sentences
3. There exists a model $\mathbb{A}$ of T such that for any finite subset $\{\varphi_1(\mathbf{x}), \ldots, \varphi_n(\mathbf{x})\} \subseteq \Sigma(\mathbf{x})$, $\mathbb{A} \models (\exists \mathbf{x})(\bigwedge \varphi_i(\mathbf{x}))$.

Examples. The set consisting of all formulas $\varphi_n(x) : (0 < x < 1/n)$ is a partial 1-type for the theory T of ordered fields. This partial 1-type is realized in an ordered field if the field has a positive infinitesimal, else it is omitted.

There is an n-type $\Sigma(v_1, \ldots, v_n)$ in the language of $\mathbb{F}$-vector spaces whose realizations in a model are the $\mathbb{F}$-linearly independent sequences of length n.

There is an 1-type $\Sigma(t)$ in the language of fields whose realizations in a model are the transcendental numbers. (I.e., numbers transcendental over the prime subfield.)

Elementary embedding/substructure/extension

An **elementary map** $j : \mathbb{A} \to \mathbb{B}$ is a type-preserving function. This means that for every $\mathbf{a} \in \mathbb{A}^n$ we have $\mathrm{tp}_{\mathbb{A}}(\mathbf{a}) = \mathrm{tp}_{\mathbb{B}}(j(\mathbf{a}))$. Equivalently, for every $\mathbf{a} \in \mathbb{A}^n$ we have $\mathbb{A} \models \varphi[\mathbf{a}]$ iff $\mathbb{B} \models \varphi[j(\mathbf{a})]$.

Most functions are not elementary maps. It is hard to find elementary maps, and hard to establish that a map is elementary. It is usually easy to show that a map is not elementary.

- The inclusion $\langle \mathbb{N}; + \rangle \hookrightarrow \langle \mathbb{Z}; + \rangle$ is not elementary. ($\mathbb{N} \not\equiv \mathbb{Z}$)
- The map $s : \langle \omega; \in \rangle \to \langle \omega; \in \rangle : n \mapsto n + 1$ is not elementary. ($\mathrm{tp}(0) \neq \mathrm{tp}(s(0))$)
- Any isomorphism is an elementary map.
- The diagonal embedding into an ultrapower is an elementary map.

Any elementary map must be injective, in fact an embedding. If the inclusion map $\mathbb{A} \to \mathbb{B}$ is elementary, we say that $\mathbb{A}$ is an **elementary substructure** of $\mathbb{B}$ ($\mathbb{A} \prec \mathbb{B}$) and that $\mathbb{B}$ is an **elementary extension** of $\mathbb{A}$ ($\mathbb{B} \succ \mathbb{A}$). In this language, $j : \mathbb{A} \to \mathbb{B}$ is elementary iff j is an embedding and $\mathrm{im}(j) \prec \mathbb{B}$.

When is $\mathbb{A} \prec \mathbb{B}$?

The Tarski-Vaught Test. Assume that $\mathbb{A}$ is a substructure of $\mathbb{B}$. TFAE:

1. $\mathbb{A} \prec \mathbb{B}$
2. Any formula with parameters in $\mathbb{A}$ that has a solution in $\mathbb{B}$ already has a solution in $\mathbb{A}$. (For every $\varphi(\mathbf{a}, y)$, if $\mathbb{B} \models (\exists y)\varphi(\mathbf{a}, y)$, then $\mathbb{A} \models (\exists y)\varphi(\mathbf{a}, y)$.)

[(1)⇒(2)] $(\exists y)\varphi(\mathbf{x}, y) \in \mathrm{tp}_{\mathbb{B}}(\mathbf{a})$ iff $(\exists y)\varphi(\mathbf{x}, y) \in \mathrm{tp}_{\mathbb{A}}(\mathbf{a})$.

[(2)⇒(1)] (Induction: atomic formulas, $\wedge$, $\neg$, $\exists$) For any embedding $e : \mathbb{A} \to \mathbb{B}$, satisfaction of atomic formulas is preserved and reflected:

$$\mathbb{A} \models \varphi[\mathbf{a}] \Leftrightarrow \mathbb{B} \models \varphi[e(\mathbf{a})].$$

This bi-implication is preserved by $\wedge$, $\neg$, so satisfaction of quantifier-free (q.f.) formulas is preserved and reflected. Even more, satisfaction of Σ_1-formulas ($\exists$(q.f.), or $\exists(\bigvee\bigwedge \pm$atomic)) are preserved. Item (2) of the theorem asserts that satisfaction of Σ_1-formulas are reflected. That's enough.

Exercises

1. What are the elementary submodels of $\langle \omega; < \rangle$?

2. If $\mathbb{F} \prec \mathbb{K}$ is a field extension that is elementary, show that any element of $\mathbb{K}$ that is algebraic over $\mathbb{F}$ lies in $\mathbb{F}$.

3. Is the field extension $\mathbb{R} \leq \mathbb{R}(t)$ elementary?

4. Show that if $\mathbb{A}, \mathbb{B} \prec \mathbb{C}$, and $A \subseteq B$, then $\mathbb{A} \prec \mathbb{B}$.

5. Give an example where $\mathbb{A}, \mathbb{B} \prec \mathbb{C}$, but $\mathbb{A} \cap \mathbb{B} \not\prec \mathbb{C}$. (Hint: Let $\mathbb{C}$ be an infinite "pure set", i.e. structure in the language of equality. Then a substructure of $\mathbb{C}$ is elementary iff it is infinite.)

Downward Lowenheim-Skolem

Thm. Let $\mathbb{B}$ be an L-structure and $X \subseteq B$ a subset. For any κ satisfying $|X| + \|L\| \leq \kappa \leq |B|$ there is an elementary substructure $\mathbb{A} \prec \mathbb{B}$ containing X which has size κ.

Proof. By enlarging X if necessary, we may assume that $|X| = \kappa$. Now define a sequence

$$X = X_0 \subseteq \mathbb{A}_0 \subseteq X_1 \subseteq \mathbb{A}_1 \cdots$$

where $\mathbb{A}_{i+1}$ is the substructure of $\mathbb{B}$ generated by X_i, and X_{i+1} is obtained from $\mathbb{A}_i$ by adjoining solutions (relative to $\mathbb{A}_i$) as needed in the Tarski-Vaught Theorem. Then $\bigcup \mathbb{A}_i = \bigcup X_i$ is a submodel of $\mathbb{B}$ since the left hand side is, while this union has the necessary solutions since the right hand side does. Let $\mathbb{A} = \bigcup \mathbb{A}_i$.

We have $\kappa \leq |X_{i+1}| \leq |A_i| + \|L\| \leq |A_i| + \kappa$ and $\kappa \leq |A_{i+1}| \leq |X_i| + \|L\| \leq |X_i| + \kappa$. Hence

$$\kappa = |X_0| \leq |\mathbb{A}| \leq |X_0| + \kappa\omega = \kappa. \square$$