Preservation theorems

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Corollary. A class $\mathcal{K}$ of structures is axiomatizable by universal sentences iff $\mathcal{K}$ is closed under ultraproducts and substructures iff $\mathbf{SP}_U(\mathcal{K}) = \mathcal{K}$.

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