Exercises about ultraproducts.

Exercise 1. Let A_n be a nonempty finite set, and assume that $|A_n| \nearrow \infty$ as $n \to \infty$. For different ultrafilters $\mathcal{U}$ on ω, what are the possible isomorphism types for $\prod_{\mathcal{U}} A_n$?

Exercise 2. Let A_n be a finite set, where $|A_n| = 2$ if n is even and $|A_n| = n$ if n is odd. What can be said about $\prod_{\mathcal{U}} A_n$ if $\mathcal{U}$ is an ultrafilter on ω?

Exercise 3. Let p_i be the i-th prime, with the enumeration starting at $i = 0$. (So $p_0 = 2, p_1 = 3$, etc.) Let $\mathbb{Z}_{p_i}$ be the cyclic group of size p_i. For different ultrafilters $\mathcal{U}$ on ω, what are the possible isomorphism types for $\prod_{\mathcal{U}} \mathbb{Z}_{p_i}$?

Exercise 4. Do the same problem as the previous one, but now treat $\mathbb{Z}_{p_i}$ as a ring instead of a group. What can you say about $\prod_{\mathcal{U}} \mathbb{Z}_{p_i}$? (Is it a field? What is its size? What is its characteristic? Can it contain $\sqrt{-1}$? Can it fail to contain $\sqrt{-1}$? Can it fail to contain $\sqrt{2}$? What about $\sqrt{3}$? What about $\sqrt{6}$? Can it fail to contain all three of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$? Is it possible that $\prod_{\mathcal{U}} \mathbb{Z}_{p_i}$ is an algebraically closed field? Can $\prod_{\mathcal{U}} \mathbb{Z}_{p_i}$ be isomorphic to $\mathbb{R}$?)

Exercise 5. Let I be a set. Let L be a language whose signature consists entirely of constant symbols, one constant symbol c_J for each subset $J \subseteq I$. For each $i \in I$, define an L-structure

$$\mathbb{A}_i = \langle \{0, 1\}; \{c_J \mid J \subseteq I\} \rangle$$

by demanding that $c_J^{\mathbb{A}_i} = 1$ if $i \in J$ and $c_J^{\mathbb{A}_i} = 0$ if $i \notin J$. Show that if $\mathcal{U} \neq \mathcal{V}$ are distinct ultrafilters in I, then $\prod_{\mathcal{U}} \mathbb{A}_i \not\equiv \prod_{\mathcal{V}} \mathbb{A}_i$.