The Completeness Theorem

$$\Sigma \models \sigma \text{ iff } \Sigma \vdash \sigma$$

Nature decides truth

The relation $\models$ defines a Galois connection between L-structures and L-sentences.

We write $\Sigma \models \sigma$ to indicate that σ lies in the Galois closure of Σ. (i.e. $\sigma \in \Sigma^{\perp\perp}$).

How can we characterize the Galois closure of Σ "internally"? (meaning: how can you determine whether $\sigma \in \Sigma^{\perp\perp}$ without referring to structures?)

Humans decide provability

We create a machine called "proof", where σ is provable from Σ $\quad (\Sigma \vdash \sigma)$ iff σ is a semantic consequence of Σ $\quad (\Sigma \models \sigma)$.

If our only goal is to characterize Galois closure internally, then we only demand that our proof calculus be

- Sound $\quad (\Sigma \vdash \sigma$ implies $\Sigma \models \sigma)$, $\quad$ and
- Complete $\quad (\Sigma \models \sigma$ implies $\Sigma \vdash \sigma)$.

But, since we are humans, we shall also demand that

- proofs should be recognizable as proofs.

Df. $\Sigma \vdash \sigma$ means there is a finite sequence of formulas

$$\alpha_1, \alpha_2, \ldots, \alpha_n = \sigma$$

where each α_i is an **axiom**, a member of Σ, or is derivable from earlier terms in the sequence using a **rule of inference**.

What is needed?

We should choose **axioms** so that they are recognizable instances of $\models \alpha$.

We should choose rules of inference, typically written $\frac{\alpha_1,\ldots,\alpha_m}{\beta}$, so that they are recognizable instances of $\{\alpha_1, \ldots, \alpha_m\} \models \beta$.

"Should" means: if we do this, then soundness will hold.

Axioms.

1. All tautologies.
2. $=$ is an equivalence relation on terms.
3. Can substitute equals for equals without changing meaning.
4. $(\forall x_i(\alpha \to \beta)) \to (\forall x_i\alpha \to \forall x_i\beta)$
5. $(\alpha \to \forall x_i\alpha)$ if x_i does not appear in formula α.
6. $(\exists x_i(x_i = t)$ if x_i does not occur in term t.

Rules.

1. (Modus Ponens) $\frac{\alpha, \alpha \to \beta}{\beta}$
2. (Generalization) $\frac{\varphi}{(\forall x_i)\varphi}$

Stage 1: the Deduction Theorem

Observe that $\Sigma \models \sigma$ iff $\Sigma \cup \{\neg\sigma\} \models \bot$. (Note: $\forall \mathbb{A}(\mathbb{A} \not\models \bot)$. I.e., $\bot$ is not satisfiable.)

Therefore we want $\Sigma \vdash \sigma$ iff $\Sigma \cup \{\neg\sigma\} \vdash \bot$.

"More generally", $\Sigma \cup \{\alpha\} \models \beta$ iff $\Sigma \models (\alpha \to \beta)$.

So we want $\Sigma \cup \{\alpha\} \vdash \beta$ iff $\Sigma \vdash (\alpha \to \beta)$.

Reverse direction is direct and easy.

Forward direction is proved by induction on the length of a proof of $\Sigma \cup \{\alpha\} \vdash \beta$.
It is also easy.
[Idea: Replace every α_i in a $(\Sigma \cup \{\alpha\})$-proof of β with $\alpha \to \alpha_i$ to obtain a Σ-proof of $(\alpha \to \beta)$.]

The second part is called:

Deduction Thm. If $\Sigma \cup \{\alpha\} \models \beta$, then $\Sigma \models (\alpha \to \beta)$.

Application

Our goal is to prove that $\Sigma \models \sigma$ implies $\Sigma \vdash \sigma$.

Equivalently, if $\Gamma := \Sigma \cup \{\neg\sigma\}$ is not **satisfiable** $(\Gamma \models \bot)$, then it is not **consistent** $(\Gamma \vdash \bot)$.

Contrapositively, if Γ is consistent, then it is satisfiable (i.e. has a model).
(This reformulation is worth remembering!)

Strategy to achieve our goal:

1. Show that a consistent theory Γ can be enlarged to a "Henkin theory".
2. Show that a Henkin theory has a model.
3. Show that a model of an enlargement of Γ is also a model of Γ. (Duh!)

Henkin theory

Df. A theory Γ is a Henkin theory if it is

1. **consistent**,
2. **complete**, and
3. **has witnesses**.

Meanings:

1. A theory is **consistent** if you can't prove falsity from it.
2. A consistent theory Γ is **complete** if it decides every sentence: For every σ, either $\sigma \in \Gamma$ or $(\neg\sigma) \in \Gamma$.
3. A theory Γ **has witnesses** if whenever $\varphi(x)$ is a formula with one free variable, then $((\exists x)\varphi(x) \to \varphi(c)) \in \Gamma$ for some constant c.

Henkin's key insight is that if $\mathbb{A}$ is a structure, then the theory of its "expansion by constants", $\Gamma = \mathrm{Th}(\mathbb{A}_A)$, is a Henkin theory. Conversely every Henkin theory arises in this way. Moreover, $\mathrm{Th}(\mathbb{A}_A)$ explains clearly how to construct its canonical model, $\mathbb{A}_A$.

The enlargement steps

Lindenbaum's Theorem. Every consistent L-theory can be enlarged to a complete L-theory.

[Idea: if $\Gamma \nvdash \sigma$, then $\Gamma \cup \{\neg\sigma\} \nvdash \bot$, so $\Gamma \cup \{\neg\sigma\}$ is a consistent enlargement of Γ. Keep doing this until you arrive at a complete theory.]

Henkin's Theorem. Every consistent theory can be enlarged to a consistent theory with witnesses, provided we allow ourselves to enlarge the language to include more constant symbols.

[Idea: suppose $\Gamma \cup \{(\exists x)\varphi(x) \rightarrow \varphi(c)\} \vdash \bot$ where $c \notin L$. Then $\Gamma \vdash \neg((\exists x)\varphi(x) \rightarrow \varphi(c))$, or $\Gamma \vdash (\exists x)\varphi(x) \wedge \neg\varphi(c)$. Need quantifier axioms and rules which permit this deduction:

$$(\exists x)\varphi(x) \wedge \neg\varphi(c), (\forall x)((\exists x)\varphi(x) \wedge \neg\varphi(x)), (\exists x)\varphi(x) \wedge \neg(\exists x)\varphi(x)), \bot.$$

Thus $\Gamma \vdash \bot$. Now repeat the idea of Lindenbaum's Theorem with σ equal to $\neg((\exists x)\varphi(x) \rightarrow \varphi(c))$.]

Finally: Henkin theories have an obvious model.

Let H be a Henkin L-theory. (= consistent, complete, with witnesses.)

Let C be the set of constants in L. It will be the domain of an L-structure.

If $c \in L$, then define $c^{\mathbb{C}} = c \in C$.

If $R(x_1, \ldots, x_n)$ is a predicate symbol, declare that $R^{\mathbb{C}}(c_1, \ldots, c_n)$ is true if $R(c_1, \ldots, c_n) \in H$.

If $F(x_1, \ldots, x_n)$ is a function symbol, declare that $F^{\mathbb{C}}(c_1, \ldots, c_n) = d$ is true if $(F(c_1, \ldots, c_n) = d) \in H$.

Define an equivalence relation θ on C by $c \equiv d \pmod{\theta}$ if $(c = d) \in H$.

It will be the case that $\mathbb{C}/\theta \models H$. In fact, $H = \mathrm{Th}(\mathbb{C}/\theta)$. $\mathbb{C}/\theta$ is called the Henkin model of H.