Cardinal and ordinal numbers.

Cardinal numbers (one, two three) are used to measure quantity, while ordinal numbers (first, second, third) are used put things in order.

Definition 1. (Ordinals)

(1) A set T is *transitive* if $R \in S \in T$ implies $R \in T$.
(2) An *ordinal (number)* is a transitive set of transitive sets.

The smallest ordinals are

$$\begin{aligned} 0 &:= \emptyset \\ 1 &:= \{0\} \\ 2 &:= \{0,1\} \\ &\vdots \\ \omega &:= \{0,1,2,\ldots\} \\ \omega+1 &:= \{0,1,2,\ldots,\omega\} \end{aligned}$$

We order ordinals by $\alpha < \beta \iff \alpha \in \beta$. Some basic properties of ordinals are

(1) (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold.
(2) Every ordinal is the set of its predecessors.
(3) There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
(4) (Well Ordering Theorem, Zermelo) Every set can be enumerated by an ordinal. (That is, for every set X there is an ordinal α and a bijection $f : \alpha \to X$.)

The Well Ordering Theorem allows us to count any set, but the ordinal α that appears in it is not unique. For example, it is clear that the identity is a bijection $f : \omega \to \omega$, but we saw in class that there is a bijection $g : \omega + 1 \to \omega$.

This non-uniqueness implies that the ordinal numbers are not appropriate for measuring size. For this we introduce cardinal numbers.

Definition 2. (Equipotence, Finiteness, Countability)

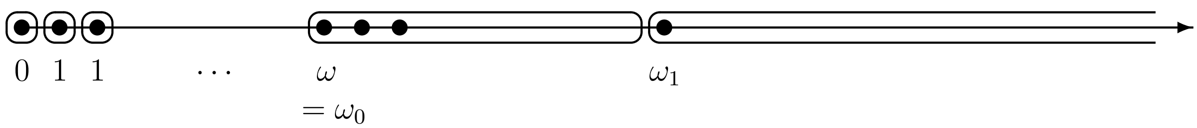
(1) $|A| = |B|$ means there is a bijection $f : A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A = |B|$ we say that A and B are *equipotent.*
(2) $|A| \leq |B|$ means there is an injection $g : A \to B$.
(3) $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
(4) X is *finite* if it is equipotent with a natural number.
(5) X is *infinite* if it is not finite.
(6) X is *countably infinite* if it is equipotent with ω.
(7) X is *countable* if it is finite or countably infinite.
(8) X is *uncountable* if it is not countable.

Theorem 3. *(Cantor-Bernstein-Schröder) If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Corollary 4. $|\mathcal{P}(\mathbb{N})| = |(0,1)| = |\mathbb{R}|$

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



The key features of this figure are

(1) Equipotence classes are intervals. The classes of natural numbers are singletons.
(2) Every equipotence class has a least element. (Such elements are called *initial ordinals.*)
(3) For every equipotence class, there is a strictly larger class.

To measure size, we pick one ordinal from each equipotence class. Since each class has a least element, that one is the natural choice.

Definition 5. A **cardinal number** is an initial ordinal.

When discussing cardinals, it is common to use the symbols $\aleph_0, \aleph_1, \aleph_2$ in place of $\omega_0, \omega_1, \omega_2$, ETC. $\aleph$ (aleph) is the first letter of the Hebrew alphabet. We read $\aleph_0$ as "aleph zero" or "aleph naught". The first few cardinals are $0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots$.

If κ is a cardinal number, then we might write $|X| = \kappa$ to mean $|X| = |\kappa|$, i.e., there is a bijection $f : \kappa \to X$. We do this even for finite cardinals, so $|X| = k$ for $k \in \mathbb{N}$ means there is a bijection $f : k \to X$.

We can refine the Well Ordering Theorem to say:

Theorem 6. *Every set can be enumerated by a unique cardinal number. (For every set X, there is a unique cardinal κ for which there is a bijection $f : \kappa \to X$.)*

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Theorem 7. *(Cantor's Theorem) If X is a set, then $|X| < |\mathcal{P}(X)|$.*