

MODEL THEORY: ASSIGNMENT 3, PROBLEM 15

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PROBLEM 15

If ZFC is consistent, then it has continuumly many countable models up to isomorphism.

Proof. Assume that ZFC is consistent. This implies that ZFC has a model. From the inequality $|L| \leq \aleph_0$ (where L is the language of ZFC) and the Löwenheim-Skolem theorem, ZFC has a model of size \aleph_0 . Now let us consider some of the potential types of any model of ZFC. Specifically we will investigate the types associated with subsets of ω . First we must show that ω is present in every model of ZFC and its elements are able to be referenced individually.

Construction of ω : We begin by defining inductive sets which leads to a natural definition of ω then we may define the elements of ω themselves. We will do so in the von Neumann style. First and foremost, we will define 0. The axiom of the empty set guarantees the following: $\exists y \forall x \neg(x \in y)$. From the axiom of extensionality, this y is unique so we may unambiguously denote it \emptyset . Let $0 = \emptyset$. Let us now define the concept of successor. Consider the formula

$$\varphi_S(x, y) := \forall z((z \in y) \leftrightarrow ((z \in x) \vee (z = x)))$$

This formula is satisfied when, and only when, $y = x \cup \{x\}$, making y the successor of x . Additionally, for any given x already shown to exist, the existence of y follows from the axioms of pairing and union. Though not shown here, it follows from extensionality that every successor is unique and distinct. Thusly, we may denote the y such that $\varphi_S(0, y)$ with the symbol 1, the z such that $\varphi_S(1, z)$ with the symbol 2, *et cetera*. Now we may define what it means for a set to be inductive. Let the predicate $I(x)$ mean that x is an inductive set. Then

$$I(X) \leftrightarrow (0 \in X \wedge \forall x(x \in X \rightarrow \forall y(\varphi_S(x, y) \rightarrow y \in X)))$$

The right half of the formula essentially states that 0 is a member of X and X is closed under the successor relation. By the axiom of infinity there exists at least one inductive set. Now we are in a perfect position to define the set ω . Let us define ω by the following formula:

$$\forall x(x \in \omega \leftrightarrow \forall y(I(y) \rightarrow x \in y))$$

This defines ω as the intersection of all inductive sets, which is itself inductive. Thus ω is manifested in every model of ZFC, since only axioms were invoked, and we are able to talk unambiguously about the elements $0, 1, 2, \dots$ which, for our purposes, will suffice.

Subsets of ω : For the moment, we will fix our model of ZFC to be \mathcal{M} . It will be shown that for each distinct subset of ω there is a unique type associated with it thus giving one potential type for each subset. Since each possible type is consistent with $Th(\mathcal{M})$, and since ZFC is a subset of $Th(\mathcal{M})$, each possible type must be consistent with ZFC itself. Let us consider a set $u \subseteq \omega$ and its one-type $tp(u) = \{\phi(x) | \mathcal{M} \models \phi(u)\}$. For all $n \in \omega$, let $\psi_n(x) = n \in x$. As an indirect consequence of the excluded middle, for every $n \in \omega$ either $\psi_n(x) \in tp(u)$ or $(\neg\psi_n(x)) \in tp(u)$. For the type of a set $u \subseteq \omega$, we have a binary option for \aleph_0 many n . This gives us 2^{\aleph_0} many distinct possible types. Since each of these types are consistent with $Th(\mathcal{M})$, they are consistent with ZFC and thus are potential types in a model of ZFC. Therefore, there are at least 2^{\aleph_0} many potential types for theories of ZFC. With this in store, all that need be done is some cardinal arithmetic and a few more deductions.

From Types To Models: We now return our attention to just the countable models. Given that there are at most ω many types per a countable model and that there are 2^{\aleph_0} many potential types, $2^{\aleph_0} \leq \omega\kappa = \kappa$, where κ denotes the number of countable models of ZFC with distinct combinations of types. Since isomorphisms preserve relations, functions, and constants, an isomorphism will preserve atomic formulas and consequently types. Thus, any two models realizing different types are non-isomorphic. So, κ is the number of non-isomorphic countable models of ZFC. Additionally, for any countable language $|L|$, such as the language of ZFC, there are at most 2^{\aleph_0} many distinct L -structures. So, $\kappa \leq 2^{\aleph_0}$. Thus, $2^{\aleph_0} \leq \kappa \leq 2^{\aleph_0}$. This directly implies $\kappa = 2^{\aleph_0}$. Therefore, there are 2^{\aleph_0} , or continuumly many, non-isomorphic countable models of ZFC. All this is assuming ZFC is consistent. \square