3. Let $n > 0$ be chosen and fixed. Let L be the language of one binary predicate, $E(x, y)$, and let T be the L-theory of one equivalence relation with n classes. That is, T is axiomatized by the sentences asserting that E defines a reflexive, symmetric, transitive binary relation, along with a sentence:

$$(\exists x_1)...(\exists x_n)\left(\left(\bigwedge_{i \neq j} \neg E(x_i, x_j)\right) \wedge (\forall y)\left(\bigvee_i E(x_i, y)\right)\right)$$

which asserts that E has exactly n classes.

The purpose of this problem is to investigate the Cantor-Bendixson rank of the closed subset $V(T) \subseteq \mathrm{Spec}(L)$ consisting of the complete theories that extend T.

(a) Describe the complete theories of Cantor-Bendixson rank 0 in $V(T)$
(b) Describe the complete theories of Cantor-Bendixson rank 1 in $V(T)$.
(c) Make a conjecture about the Cantor-Bendixson rank of $V(T)$.

Proof. (a) The relation E has n classes, each of which must contain at least one element, since an empty class would mean we actually have less than n classes.

Since the language is countable, each structure in the language is elementarily equivalent to a countable structure. Thus, any model of T is elementarily equivalent to a countable set containing an equivalence relation with exactly n classes. We will denote such a structure by listing each of its class sizes in increasing order: $(s_1, \ldots, s_n)$. Here, each s_i is a positive natural number or ω, and we may assume that $s_1 \leq \cdots \leq s_n$. We write $E(s_1, \ldots, s_n)$ to refer to the equivalence relation of this type, $\mathbb{A}(s_1, \ldots, s_n)$ to refer to the structure of this type, and $T(s_1, \ldots, s_n)$ to refer to the theory of the stucture of this type.

Claim: If all s_i are finite, then $T(s_1, \ldots, s_n)$ is isolated.

Proof: If all s_i are finite, then $\mathbb{A}(s_1, \ldots, s_n)$ is finite. Thus $\mathbb{A}(s_1, \ldots, s_n)$ is a finite structure in a finite language. By modth1p10, there is a single sentence that axiomatizes $\mathbb{A}(s_1, \ldots, s_n)$, hence $T(s_1, \ldots, s_n)$ is isolated.

Claim: If $T(s_1, \ldots, s_n)$ is isolated, then all s_i are finite.

This will be shown by contradiction. Assume that $T(s_1, \ldots, s_n)$ is isolated, but that some of our s_i are infinite. That is, assume that the type of equivalence relation is

$$(s_1, \ldots, s_k, \omega, \omega, \ldots, \omega).$$

Here there are k finite positive numbers $s_1 \leq \cdots \leq s_k$ followed by $n - k$ copies of ω.

Stage 1. There is a single sentence σ describing the finite part of the tuple $(s_1, \ldots, s_n)$. This sentence says that there exist k elements $x_1, \ldots, x_k$, that x_i and x_j are not E-equivalent when $i \neq j$, and that the number of elements E-equivalent to x_i is s_i.

Stage 2. $\mathbb{A}(s_1,\ldots,s_n)$ is axiomatized by the following set of sentences, $\tau_1,\tau_2,\tau_3,\ldots$ where:

τ_N expresses that the finite part of $(s_1,\ldots,s_n)$ is described by σ, and any element not in one of the classes of size $s_1,\ldots,s_k$ belongs to a class of size bigger than N and bigger than $s_1,\ldots,s_k$. That is, any countable model of $\{\tau_N|N \in \omega\}$ has the correct finite part, and all other classes must have size ω, hence the only countable model of this set, up to isomorphism, is $\mathbb{A}(s_1,\ldots,s_n)$.

Stage 3. (Obtaining the contradiction.) If $T(s_1,\ldots,s_n)$ is isolated, then it is axiomatizable by a single sentence. Since it is also axiomatizable by the set $\{\tau_N|N \in \omega\}$, the Compactness Theorem guarantees that it is axiomatizable by a finite subset of $\{\tau_N|N \in \omega\}$, and therefore by a single sentence τ_N (since they get increasingly weaker). But it is easy to see that $T(s_1,\ldots,s_n)$ is not axiomatizable by a single τ_N, since it is possible to build a finite structure satisfying any given sentence τ_N and there are no finite models of $T(s_1,\ldots,s_n)$.

(b) Let $T'(s_1,\ldots,s_n)$ refer to the theory of those structures with at least one infinite class. T' is obtained from T by adding all sentences of the form $(\neg\sigma)$ where σ is as defined in Part (a) Stage 1. Now, to determine the complete theories extending T of CB rank 1 in $V(T)$, one only needs to determine the complete theories extending T' of CB rank 0 in $V(T')$.

Claim. If all s_i with $i < n$ are finite, then $T(s_1,\ldots,s_{n-1},\omega)$ is isolated in $V'(T)$.

As in Part (a), we may identify the complete theories by examining countable models of T'. Each such model has the form $\mathbb{A}(s_1,\ldots,s_{n-1},\omega)$, where $s_1 \leq \cdots \leq s_{n-1} \leq \omega$. If $s_1,\ldots,s_{n-1}$ are all finite, then it follows from the Part (a) Stage 1 that $\mathbb{A}(s_1,\ldots,s_{n-1},\omega)$ is finitely axiomatizable relative to T'. Hence $T(s_1,\ldots,s_{n-1},\omega)$ is isolated in $V(T')$ and hence $T(s_1,\ldots,s_{n-1},\omega)$ has CB rank 1 in $V(T)$.

Claim. If $T(s_1,\ldots,s_{n-1},\omega)$ is isolated, then all s_i, $i < n$, are finite.

This will be a proof by contradiction. Assume that $T(s_1,\ldots,s_{n-1},\omega)$ is isolated in $V(T')$, but that some of the s_i, $i < n$, are infinite. That is, assume that the type of equivalence relation is

$$(s_1,\ldots,s_k,\omega,\omega,\ldots,\omega).$$

Here there are k finite positive numbers $s_1 \leq \cdots \leq s_k$ followed by $n - k$ copies of ω, where $n - k > 1$.

Stage 1. We have already shown in Part (a) there is a single sentence σ describing the finite part of the tuple $(s_1,\ldots,s_n)$.

Stage 2. $\mathbb{A}(s_1,\ldots,s_n)$ is axiomatized by the following set of sentences, $\tau_1,\tau_2,\tau_3,\ldots$ where:

τ_N expresses that the finite part of $(s_1,\ldots,s_n)$ is described by σ, and any element not in one of the classes of size $s_1,\ldots,s_k$ belongs to a class of size bigger that N and bigger than $s_1,\ldots,s_k$. That is, any countable model of $\{\tau_N|N \in \omega\}$ has the correct finite part, and

all other classes must have size ω, hence the only model of this set, up to isomorphism, is
$\mathbb{A}(s_1, \ldots, s_n)$.

Stage 3. (Obtaining the contradiction.) If $T(s_1, \ldots, s_n)$ is isolated, then it is axiomatizable in $V(T')$ by a single sentence. Since it is also axiomatizable by the set $\{\tau_N | N \in \omega\}$, the Compactness Theorem guarantees that it is axiomatizable by a finite subset of $\{\tau_N | N \in \omega\}$, and therefore by a single sentence τ_N (since they get increasingly weaker). But it is easy to see that $T(s_1, \ldots, s_n)$ is not axiomatizable by a single τ_N, since it is possible to build a structure satisfying any given sentence τ_N which has exactly one infinite class.

(c) For each additional infinite class we increase the Cantor-Bendixson rank of its complete theory by one. Our maximal case is when all of our classes have cardinality ω, $(\omega, \omega, \ldots, \omega)$, whose complete theory will only become isolated after the nth Cantor-Bendixson derivative. Therefore we hypothesize that the Cantor-Bendixson rank of $V(T) = $ n.

$\square$