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Proof. the class of well ordered sets is not elementary

Lemma. *A linear ordered set is Well Ordered IFF it does not contain an infinite and descending sequence.*

Let the language of well ordered sets be $\mathscr{L} = \{<\}$.

Suppose for the sake of contradiction some theory $\mathscr{T}$ axiomatizes the class of well ordered sets.

Expand the language $\mathscr{L}$ to $\mathscr{L}_2$ by including the constants $\{a_1, a_2, \ldots\}$. Consider the set of sentences:
$S_1 := \{(a_m > a_n) \big| n > m\}$.

Let $\mathscr{T}_2 = \mathscr{T} \cup S_1$. We will show that any finite subset of this theory is satisfiable. Let $\mathscr{T}_f$ be an arbitrary finite subset of $\mathscr{T}_2$. $\mathscr{T}_f$ will contain a finite number of sentences in S_1, $\{s_{i_1}, \cdots, s_{i_k}\}$. Let W be the finite set of constants that appear in $\{s_{i_1}, \cdots, s_{i_k}\}$. The Sentences in S_1 define an order on W as a finite descending chain. This set is a well ordered set so W satisfies $\mathscr{T}_f$. Therefore W satisfies $\mathscr{T}$ and $\{s_{i_1}, \cdots, s_{i_k}\}$.

Because every finite subset of $\mathscr{T}_2$ is satisfiable, by the compactness theorem $\mathscr{T}_2$ is satisfiable. However the model of $\mathscr{T}_2$ is a well ordered set with an infinite descending chain and a well ordered set cannot contain an infinite descending chain, thus contradicting the initial assumption that there exists a theory of well ordered sets.

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