

# modth3p6

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**Problem 6.** Suppose that a theory  $T$  in the language of countably many unary relations  $R_1, R_2, \dots$  contains every sentence of the form

$$\exists x(R_{i_1}(x) \wedge \dots \wedge R_{i_m}(x) \wedge \neg R_{j_1}(x) \wedge \dots \wedge \neg R_{j_n}(x))$$

where  $i_1, \dots, i_m, j_1, \dots, j_n$  are distinct.

- (a) Show that  $T$  has quantifier elimination and is complete.
- (b) Derive from (a) that any  $n$ -type is generated by formulas of the form  $\pm$ atomic, that is, those of the form  $x_i = x_j$ ,  $x_i \neq x_j$ ,  $R_i(x_j)$ , and  $\neg R_i(x_j)$ .
- (c) Explain why  $S_n(T)$  has no isolated points, hence is homeomorphic to the Cantor set.

*Proof.* Words.

- (a) First we will prove that if  $\phi$  is a quantifier-free formula, then for some quantifier free formula  $\psi$ ,

$$T \models (\exists x_i \phi) \leftrightarrow \psi.$$

In disjunctive normal form  $\phi$  can be written as

$$\bigvee_i \bigwedge_j \phi_{i,j}$$

where each  $\phi_{i,j}$  is an atomic formula or the negation of an atomic formula. But since  $\exists$  distributes over disjunction, it will be sufficient to prove the claim for a conjunction of plus-minus atomic formulas

$$\phi = \bigwedge_i \phi_i.$$

We may assume, without loss of generality, that  $\phi$  contains no atomic formulas of the form  $x_k = x_k$  or  $x_k \neq x_k$  since the first condition can be dropped and the second condition reduces  $\phi$  to any formula that is always false. For more or less the same reasons, we may assume that  $\phi$  does not reference the same relation-variable pair twice. That is, we may assume it does not contain  $R_p(x_q) \wedge R_p(x_q)$ ,  $\neg R_p(x_q) \wedge \neg R_p(x_q)$  or  $R_p(x_q) \wedge \neg R_p(x_q)$ . Since there are no constants or functions in the language, all plus-minus atomic formulas are of the form  $x_i = x_j$ ,  $x_i \neq x_j$ ,  $R(x_i)$ , or  $\neg R(x_i)$ . If  $\phi$  contains an atomic formula of the form  $x_k = x_l$  for  $k \neq l$  then we have that

$$T \models \exists x_k \phi(x_1, \dots, x_k, \dots, x_l, \dots, x_n) \leftrightarrow \phi(x_1, \dots, x_l, \dots, x_l, \dots, x_n).$$

However, if  $\phi$  does not contain an atomic formula of that form, then it is possible to write  $\phi$  as

$$R_{i_1}(x_k) \wedge \dots \wedge R_{i_m}(x_k) \wedge \neg R_{j_1}(x_k) \wedge \dots \wedge \neg R_{j_n}(x_k) \wedge (x_k \neq x_{l_1}) \wedge \dots \wedge (x_k \neq x_{l_o}) \wedge \pi(x_1, \dots, \hat{x}_k, \dots, x_n),$$

for some formula  $\pi$  that does not reference  $x_k$ . If  $\pi$  is empty, then we may assume  $\pi = \top$ . Now because any model of  $T$  must have infinitely many elements that satisfy any combination of distinct relations and their negations, we have

$$T \models \exists x_k \phi \leftrightarrow \pi(x_1, \dots, \hat{x}_k, \dots, x_n),$$

proving the claim.

The theory  $T$  satisfies the conditions of Lemma 3.1.5 (Marker), so it follows that  $T$  has quantifier elimination. Now let  $\phi$  be some sentence. In a given model, the truth of  $\phi$  is independent of its valuation. We know that

$$T \models \phi \leftrightarrow \psi(x)$$

for some quantifier-free one-variable formula  $\psi$  which is independent of its valuation. Since  $\psi(x)$  takes on the same value regardless of  $x$ , we have

$$T \models \psi(x) \leftrightarrow \exists x \psi(x)$$

and thus

$$T \models \phi \leftrightarrow \exists x \psi(x).$$

By a reduction similar to that used to show quantifier elimination, where our current variable plays the role of  $x_k$ , the formula  $\exists x \psi(x)$  is logically decided by  $T$ , since it is a disjunction over conjunctions that are either entailed by the axioms, or refuted because they contain a term of the form  $x \neq x$ . Therefore  $T$  must be complete.

- (b) Let  $p \in S_n(T)$  be some  $n$ -type. Let  $D = \{\phi \in p : \phi \text{ is a plus-minus atomic formula}\}$ . The notation  $\langle D \rangle$  will signify the partial  $n$ -type generated by  $D$ . We claim that  $p = \langle D \rangle$ . Clearly  $\langle D \rangle \subset p$ , so it is sufficient to prove that  $p \subset \langle D \rangle$ . Suppose that  $\psi \in p$  is a conjunction of plus-minus atomics. That is, suppose

$$\psi = \psi_1 \wedge \dots \wedge \psi_n \in p,$$

where  $\psi_i$  is atomic or the negation of an atomic formula for  $1 \leq i \leq n$ . Then for each index  $i$ , we must have  $\psi_i \in p$  which implies  $\psi_i \in D$  which further shows that  $\psi \in \langle D \rangle$ . Now suppose that  $\eta = \eta_1 \vee \dots \vee \eta_m \in p$ , where each  $\eta_i$  is the conjunction of plus-minus atomics. Because  $p$  is a complete type, we have that  $\psi_i \in p$  with  $1 \leq i \leq m$  for at least one index  $i$ . Since  $\psi_i$  is the conjunction of plus-minus atomics it follows that  $\psi_i \in \langle D \rangle$  for some  $i$ . Therefore  $\psi \in \langle D \rangle$ . Finally, suppose that  $\phi \in p$  is some formula. Because quantifier elimination holds,

$$T \models \phi(x_1, \dots, x_n) \leftrightarrow \lambda(x_1, \dots, x_n)$$

for some quantifier-free formula  $\lambda$ . Clearly  $\lambda \in p$ , showing by the previously established results that  $\lambda \in \langle D \rangle$ . That gives us  $\phi \in \langle D \rangle$  as desired.

- (c) The isolated  $n$ -types are exactly the  $n$ -types isolated by a single formula. But no complete  $n$ -type can be defined by a single formula. Suppose there was some formula  $\phi(x_1, \dots, x_n)$  that isolated a complete  $n$ -type. Then, by the result proven in part (a), that complete  $n$ -type must be isolated by some quantifier-free formula  $\psi(x_1, \dots, x_n)$ . Because  $\phi$  mentions only finitely many relations, there must be some relation  $R_k$  that is not referenced by the formula  $\phi$ . But since  $\langle \phi \rangle$  is complete, either  $R_k(x_1) \in \langle \phi \rangle$  or  $\neg R_k(x_1) \in \langle \phi \rangle$ . If the former holds, then clearly

$$\phi \wedge R_k(x_1)$$

is satisfiable and has some model  $\mathcal{M}$ . Define  $\overline{\mathcal{M}}$  to be an  $\mathcal{L}$ -structure identical to  $\mathcal{M}$  except for the fact that  $R_k^{\overline{\mathcal{M}}} = (R_k^{\mathcal{M}})^c$ . For some  $x$  in the universe  $M$ ,  $\mathcal{M} \models R_k(x)$  if and only if  $\overline{\mathcal{M}} \models \neg R_k(x)$ . Looking back at the axioms of  $T$ , it can be seen that  $\overline{\mathcal{M}} \models T$ . Also, since  $\phi$  does not reference  $R_k$ , we have  $\overline{\mathcal{M}} \models \phi$ . The existence of  $\overline{\mathcal{M}}$  proves that the formula

$$R_k(x_1) \wedge \neg R_k(x_1)$$

is satisfiable. But that shows that neither  $\neg R_k(x_1)$  nor  $R_k(x_1)$  can be part of the type generated by  $\phi$ , a contradiction of our assumption that  $\phi$  generates an  $n$ -type. If instead we had assumed that  $\neg R_k(x_1) \in \langle \phi \rangle$ , a similar argument would generate a contradiction.

It follows that  $S_n(T)$  has no isolated points. As a perfect Stone space with a countable basis,  $S_n(T)$  must be homeomorphic to the Cantor set.

□