

13. Show that any structure is embeddable in an ultraproduct of its finitely generated substructures. Conclude that any universal class is generated by its finitely generated members. Show that this statement about universal classes is not true for arbitrary elementary classes.

Proof. Consider a structure $\mathbb{A} = \langle A; R^{\mathbb{A}}, F^{\mathbb{A}}, c^{\mathbb{A}} \rangle$, and let \mathcal{F} be the set of finitely generated substructures of \mathbb{A} . Take subsets of \mathcal{F} of the form $X_{\mathbb{F}} := \{\mathbb{F}' \in \mathcal{F} \mid \mathbb{F} \subseteq \mathbb{F}'\}$, then extend the set $\{X_{\mathbb{F}} \mid \mathbb{F} \in \mathcal{F}\}$ to an ultrafilter \mathcal{U} on \mathcal{F} .

We now wish to define an embedding of \mathbb{A} into the ultraproduct $\prod_{\mathcal{U}} \mathbb{F}$. For each \mathbb{F} , choose some auxiliary element $x^{\mathbb{F}} \in \mathbb{F}$, then define $\iota : \mathbb{A} \rightarrow \prod_{\mathbb{F} \in \mathcal{F}} \mathbb{F}$ by

$$\iota(a) = (a^{\mathbb{F}})_{\mathbb{F} \in \mathcal{F}}$$

where $a^{\mathbb{F}} = a$ if $a \in \mathbb{F}$ or $a^{\mathbb{F}} = x^{\mathbb{F}}$ otherwise. Let $\pi : \prod_{\mathbb{F} \in \mathcal{F}} \mathbb{F} \rightarrow \prod_{\mathcal{U}} \mathbb{F}$ denote the quotient map by the equivalence relation $\theta_{\mathcal{U}}$. Suppose we had instead constructed the map ι using a different choice of $x^{\mathbb{F}}$ for each \mathbb{F} — denote this by ι' — then we would have

$$\llbracket \iota(a) = \iota'(a) \rrbracket \supseteq \{\mathbb{F} \in \mathcal{F} \mid a \in \mathbb{F}\} = \{\mathbb{F} \in \mathcal{F} \mid \mathbb{F}_a \subseteq \mathbb{F}\} = X_{\mathbb{F}_a}$$

where \mathbb{F}_a is the substructure of \mathbb{A} generated by the element a . Hence $\llbracket \iota(a) = \iota'(a) \rrbracket \in \mathcal{U}$ and so $\pi \circ \iota$ does not depend on the choice of $x^{\mathbb{F}}$, however we still need to show that $\pi \circ \iota$ is an embedding. Consider a formula of the form $\pm R(x_1, \dots, x_n)$ where R is a relation in $R^{\mathbb{A}}$ (we can assume that \mathbb{A} is purely relational, since relations can simulate functions and constants). Then for any tuple $(a_1, \dots, a_n) \in A^n$ where $\pm R(a_1, \dots, a_n)$ is *not* true, the set

$$\llbracket \pm R(\iota(a_1), \dots, \iota(a_n)) \rrbracket$$

cannot contain $X_{\mathbb{F}}$ for any \mathbb{F} . This is because there exists an $\mathbb{F}' \supseteq \mathbb{F}$ containing a_1, \dots, a_n , and $\mathbb{F}' \in \llbracket \pm R(\iota(a_1), \dots, \iota(a_n)) \rrbracket$ would imply that $\pm R(a_1^{\mathbb{F}'}, \dots, a_n^{\mathbb{F}'})$ is true. However, this is a contradiction since we took \mathbb{F}' to contain each a_i , and so $a_i^{\mathbb{F}'} = a_i$ for each i and we assumed $\pm R(a_1, \dots, a_n)$ is false. Thus $\pi \circ \iota$ is injective (take R to be the equality relation), but moreover it is an embedding of \mathbb{A} into $\prod_{\mathcal{U}} \mathbb{F}$ since ι preserves all relations and their negations.

A universal class is a class axiomatized by universally quantified sentences, that is, sentences of the form $\forall x \varphi(x)$ which do not contain existential quantifiers. Let K be a universal class. Note that if a universally quantified sentence σ is satisfied by $\mathbb{A} \in K$, it will necessarily be satisfied by any of its substructures, hence $\mathbb{A} \in K$ implies that $\mathbb{F} \in K$ for any substructure \mathbb{F} of \mathbb{A} . Conversely, if $\mathbb{F}_i \models \sigma$ for each \mathbb{F}_i — where $\{\mathbb{F}_i\}_{i \in I}$ is the set all finitely generated substructures of \mathbb{A} — then $\prod_{\mathcal{U}} \mathbb{F}_i \models \sigma$, but since \mathbb{A} embeds into $\prod_{\mathcal{U}} \mathbb{F}_i$ as shown above, we can conclude that $\mathbb{A} \in K$, and so any universal class is generated by its finitely generated members.

However, for example, let K be the class of infinite pure sets. That is, K is the class of infinite structures in the language of equality. This class is elementary and non-empty, but contains no finitely generated substructures (since a finitely generated set is finite), thus K cannot be generated by its finitely generated members.

□