

MODEL THEORY HOMEWORK 1

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Problem (8). Show that the class of connected graphs is not elementary.

We work with the language of graphs $\mathcal{L} = \{E\}$ where E is a binary relation symbol. Our theory will be that of undirected graphs without self loops or multiple edges, axiomatized by the declaring that E is irreflexive and symmetric:

- $\forall x \neg E(x, x)$
- $\forall x \forall y (E(x, y) \rightarrow E(y, x))$

A graph G is connected if for every pair of distinct vertices $x, y \in G$, either $x = y$ or there exists a path from x to y , i.e. a finite sequence $\{v_1, \dots, v_n\}$ of *distinct* vertices with $v_1 = x$, $v_n = y$, and $E^G(v_i, v_{i+1})$ for every $1 \leq i < n$.

Theorem. *The class of connected graphs is not an elementary class.*

Proof. Suppose toward contradiction that Σ was the theory of connected graphs (extending the theory of undirected graphs). Extend \mathcal{L} by two constants, so that $\mathcal{L}' = \mathcal{L} \cup \{a, b\}$. For each n , let $p_n(x, y)$ be a formula declaring that there is a no path of length n between a and b :

$$p_n = \neg \left(\exists v_1 \dots \exists v_n \left((v_1 = a) \wedge \bigwedge_{i=1}^{n-1} E(v_i, v_{i+1}) \wedge (v_n = b) \wedge \bigwedge_{i \leq j} v_i \neq v_j \right) \right)$$

Note as explained above that we are defining paths to be “proper” in the sense that they consist entirely of distinct vertices, and p_1 means $a \neq b$.

Now let Σ' be the \mathcal{L}' -theory $\Sigma \cup \{p_n : n \in \mathbb{N}\}$. Any finite subset of Σ' is satisfiable: Indeed, for any such subset there must be a $k \in \mathbb{N}$ for which it does not contain p_i for $i \geq k$, and as such it is modeled by the k -chain $\mathbb{G} = (G, E^{\mathbb{G}}, a^{\mathbb{G}}, b^{\mathbb{G}})$ where $G = \{1, \dots, k\}$, $E^{\mathbb{G}}(i, j) \Leftrightarrow i = j + 1$ or $j = i + 1$, $a^{\mathbb{G}} = 1$ and $b^{\mathbb{G}} = k$; since this graph is connected it would also satisfy any sentences in Σ . Thus, by compactness, Σ' is satisfiable. A graph \mathbb{G} that models Σ' has a pair of vertices with no path of finite length between them (that is, no path between them), but \mathbb{G} satisfies Σ so it must be connected; a contradiction. \square