MATH 6000 - Assignment 1

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1. Show that the following pairs of abelian groups are not elementarily equivalent.

   (a) $\mathbf{Z}$ and $\mathbf{Q}$.

   Proof. Let $\mathcal{L} = \{+, 0\}$ be the language of abelian groups. Then the $\mathcal{L}$-sentence

   $$\forall x \exists y (x = y + y)$$

   is true in $\mathbf{Q}$ but not in $\mathbf{Z}$. If $\frac{p}{q} \in \mathbf{Q}$, then we have $\frac{p}{2q} \in \mathbf{Q}$ with $\frac{p}{q} = \frac{p}{2q} + \frac{p}{2q}$. However, $1 \in \mathbf{Z}$ is not equal to $y + y$ for any integer y since 1 is not even. Therefore, $\mathbf{Z}$ and $\mathbf{Q}$ are not elementarily equivalent. ■

   (b) $\mathbf{Z}$ and $\mathbf{Z} \times \mathbf{Z}$.

   Proof. Using the same language as in part (a), the $\mathcal{L}$-sentence

   $$\exists m \forall n \exists \ell \left((n = \ell + \ell) \vee (n = \ell + \ell + m) \right)$$

   is true in $\mathbf{Z}$ but not in $\mathbf{Z} \times \mathbf{Z}$. Take $1 \in \mathbf{Z}$ and fix $n \in \mathbf{Z}$. If n is even, then $n = \frac{n}{2} + \frac{n}{2}$. If n is odd, then $n = \frac{n-1}{2} + \frac{n-1}{2} + 1$. We will prove the $\mathcal{L}$-sentence above does not hold in $\mathbf{Z} \times \mathbf{Z}$; that is, we will prove

   $$\forall (a, b) \exists (m, n) \forall (q, r) \left(((m, n) \neq (2q, 2r)) \wedge (m, n) \neq (2q + a, 2r + b) \right) .$$

   Fix $(a, b) \in \mathbf{Z} \times \mathbf{Z}$.
   Case I: If a is even, then consider $(1, 0) \in \mathbf{Z} \times \mathbf{Z}$ and fix $(q, r) \in \mathbf{Z} \times \mathbf{Z}$. Then $(1, 0) \neq (2q, 2r)$ and $(1, 0) \neq (2q + a, 2r + b)$ since $2q$ and $2q + a$ are both even while 1 is odd.
   Case II: If a is odd, then consider $(0, 1) \in \mathbf{Z} \times \mathbf{Z}$ and fix $(q, r) \in \mathbf{Z} \times \mathbf{Z}$. Then $(0, 1) \neq (2q, 2r)$ since $2r$ is even while 1 is odd, and $(0, 1) \neq (2q + a, 2r + b)$ since $2q + a$ is odd while 0 is even. Therefore, $\mathbf{Z}$ and $\mathbf{Z} \times \mathbf{Z}$ are not elementarily equivalent. ■