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6. Let L be a language in a finite signature. Then there are at least $\aleph_0$-many distinct closed L-theories, and at most $2^{\aleph_0}$-many closed distinct theories.

Proof. Without loss of generality, we will assume that our signature contains only relation symbols, since we can regard functions and constants as special instances of these.

To see that there are at minimum $\aleph_0$-many theories, consider for each $n \in \mathbb{Z}^+$ the sentence $\sigma_n \coloneqq \exists x_1 \ldots \exists x_n (\bigwedge_{i<j\leq n} x_i \neq x_j)$. Then consider $\phi_n = \sigma_n \wedge \neg\sigma_{n+1}$. This sentence is true in a model when it has exactly n elements, hence there are at least $\aleph_0$ many models.

Now we show that there are at most $2^{\aleph_0}$ many theories. Since L is a language in a finite signature, it has size $\aleph_0$. A theory consists of a particular subset of sentences from L, hence is a subset of the powerset of L. As $|\mathcal{P}(L)| = 2^{\aleph_0}$, any subset of it has cardinality no more than $2^{\aleph_0}$, so in particular this is true for theories. ■