MODEL THEORY
HOMEWORK ASSIGNMENT III

Read Chapter 3.

PROBLEMS

1. (Toby Aldape, Hayden Hollis, Mateo Muro)
You might conjecture that the composition of elementary embeddings is elementary, or that the composition of nonelementary embeddings is nonelementary. This problem explores all relationships of this type. Namely it explores which maps, in a triple of embeddings $(f, g, g \circ f)$, can be (or are forced to be) elementary.
For each triple $(x, y, z) \in \{\mathbf{elementary}, \mathbf{not}\}^3$ find an example

$$\mathbb{A} \xrightarrow{f} \mathbb{B} \xrightarrow{g} \mathbb{C}$$

realizing the triple, if possible, or explain why there is no example.

2. (Chase Meadors, Connor Meredith, Andrew Stocker)
Prove that if L-structures $\mathbb{A}$ and $\mathbb{B}$ are elementarily equivalent, then there exists an L-structure $\mathbb{C}$ into which both can be elementarily embeded.
[Hint: There is no harm in assuming that $A \cap B = \emptyset$. Then show that $\mathbf{Th}(\mathbb{A}_A) \cup \mathbf{Th}(\mathbb{B}_B)$ is a consistent $L(A \cup B)$-theory.]

3. (Chris Eblen, Vishnu Murali, Joel Ornstein)
Suppose that L is a language and L' is an expansion of L by some set C of additional constant symbols. Suppose that T is an L-theory that has quantifier elimination and that $T' \supseteq T$ is an L'-theory extending T. Show that T' has q.e.
The theory of dense linear order without endpoints has q.e. (you may assume this). Show that any theory of dense linear order with some additional constant symbols is complete iff the theory completely decides how the order relation restricts to the interpretations of the constant symbols.

4. (Howie Jordan, Thomas Magnuson, Patrick Wynne)
A structure is *ultrahomogeneous* if every isomorphism between finitely generated substructures extends to an automorphism. Show that if $\mathbb{A}$ is

a finite L-structure, then Th($\mathbb{A}$) has quantifier elimination iff $\mathbb{A}$ is ultrahomogeneous.

5. (Raymond Baker, Sangman Lee, Trevor Manders)
An L-theory T is *substructure complete* if whenever $\mathbb{A} \leq \mathbb{B}$ and $\mathbb{B} \models T$, then $T \cup$ Diag($\mathbb{A}$) axiomatizes a complete $L(A)$-theory. Prove that T has quantifier elimination iff it is substructure complete.

6. (Toby Aldape, Hayden Hollis, Mateo Muro)
The theory T of countably many independent unary relations is the theory in the language with relation symbols $R_n(x)$, $n < \omega$, which contains all sentences of the form

$$\exists x(R_{i_1}(x) \wedge \cdots \wedge R_{i_m}(x) \wedge \neg R_{j_1}(x) \wedge \cdots \wedge \neg R_{j_n}(x))$$

whenever $i_1, \ldots, i_m, j_1, \ldots, j_n$ are distinct.

(a) Show that T has quantifier elimination and is complete.
(b) Derive from (a) that any n-type is generated by formulas of the form $\pm$atomic, i.e., those of the form $x_i = x_j$, $x_i \neq x_j$, $R_i(x_j)$, and $\neg R_i(x_j)$.
(c) Explain why $S_n(T)$ has no isolated points, hence is homeomorphic to the Cantor set.

7. (Chase Meadors, Connor Meredith, Andrew Stocker)
Let L be the language whose only nonlogical symbol is one unary relation symbol. Find all L-theories that have quantifier elimination. Are they all complete?

8. (Chris Eblen, Vishnu Murali, Joel Ornstein)
Which finite abelian groups A have the property that Th(A) has quantifier elimination?
[Hint: Make some use of Problem 4. Prove that a finite abelian group is ultrahomogeneous iff its Sylow subgroups are ultrahomogeneous.]

9. (Howie Jordan, Thomas Magnuson, Patrick Wynne)
Which finite, simple, unital rings R have the property that Th(R) has quantifier elimination?
[Hints: Use the Wedderburn-Artin Theorem and Wedderburn's Little Theorem to show that $R \cong M_n(\mathbb{F}_q)$ for some positive integer n and some prime power q. To complete the problem you must find the allowable values of n and q. It may help to know that every automorphism of R is inner (see the Skolem-Noether Theorem).]

10. (Raymond Baker, Sangman Lee, Trevor Manders)
In this problem a *graph* is a structure $G = \langle V; E\rangle$ where E is a binary, irreflexive, symmetric relation on V.

(a) Classify the finite disconnected graphs G such that Th(G) has quantifier elimination.
(b) Classify the finite graphs G with disconnected complement such that Th(G) has quantifier elimination.
(c) It is known that every finite graph G where Th(G) has quantifier elimination is of the type in part (a) or part (b) or else is one of the following two graphs: (i) the 5-element cycle, and (ii) the line graph of $K_{3,3}$. Draw pictures of these two graphs. (Include the definitions of $K_{3,3}$ and *line graph* in your solution.)

11. (Toby Aldape, Hayden Hollis, Mateo Muro)
Let $t(n)$ be the least number N such that there exists some infinite structure with exactly N n-types over the empty set. (So $t(1) = 1, t(2) = 2, t(3) = 5, \ldots$.) Show that there exists a single infinite structure with exactly $t(n)$ n-types for each n. Classify all such structures.

12. (Chase Meadors, Connor Meredith, Andrew Stocker)
Let T be a theory in a countable language. Show that if T has an infinite model, then some countable model of T is not finitely generated.
[Hint: Use elementary chains.]

13. (Chris Eblen, Vishnu Murali, Joel Ornstein)
Let T be the theory of $\mathbb{A} = \langle \omega; \cdot, +\rangle$, and show that $|S_1(T)| = 2^\omega$. Conclude that there are 2^ω countable models of T up to isomorphism.

14. (Howie Jordan, Thomas Magnuson, Patrick Wynne)
Let $\mathbb{Q}$ be the field of rational numbers. Show that Th$(\mathbb{Q})$ has continuumly many countable models up to isomorphism.

15. (Raymond Baker, Sangman Lee, Trevor Manders)
Prove that, if ZFC is consistent, then it has continuumly many countable models up to isomorphism.