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7. Show that the following classes of graphs are elementary:

(a) The class of infinite graphs.

(b) The class of triangle-free graphs.

(c) The class of 2-colorable graphs.

Proof. An elementary class is a class of L-structures that are definable by a set of L-sentences. We work with the language of graphs L = {R} where R is a binary relation symbol. The theory of graphs T is axiomatized by declaring that R is irreflexive and symmetric:

- $\forall$x $\neg$R(x,x)
- $\forall$x$\forall$y R(x,y) $\to$ R(y,x)

(a) The class of infinite graphs.

If we extend T such that T' = T $\cup$ $\{\phi_n : n \in \omega\}$ where $\phi_n = \exists v_1 \exists v_2 ... \exists v_n (\bigwedge_{i<j\leq n} v_i \neq v_j)$, then T' defines the class of infinite graphs. Hence it is elementary.

(b) The class of triangle-free graphs.

If we extend T such that T' = T $\cup$ $\{\tau\}$ where $\tau = \forall v_1 \forall v_2 \forall v_3 (\mathrm{R}(v_1, v_2) \wedge \mathrm{R}(v_2, v_3) \to \neg \mathrm{R}(v_3, v_1))$, then T' defines the class of triangle-free graphs. Hence it is elementary.

(c) The class of 2-colorable graphs.

Here we will use the fact that a graph is 2-colorable if and only if it contains no odd cycles. Thus, if we extend T such that T' = T $\cup$ $\{\phi_n : 2 \leq n \leq \omega\}$ where $\phi_n = \forall v_1 \forall v_2 ... \forall v_{2n-1} (\mathrm{R}(v_1, v_2) \wedge \mathrm{R}(v_2, v_3) \wedge ... \wedge \mathrm{R}(v_{2n-2}, v_{2n-1}) \to \neg \mathrm{R}(v_{2n-1}, v_1))$, then T' defines the class of 2-colorable graphs. Hence it is elementary.

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