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Let $t(n)$ be the least number N such that there exists some infinite structure with exactly N n-types over the empty set. We will show there exists a single infinite structure with exactly $t(n)$ n-types for each n and classify these structures.

Pf. For an equivalence relation E on $\{1, \ldots, n\}$, define $\varepsilon_E(x_1, \ldots, x_n)$ to be the conjunction of all atomic formulas $(x_i = x_j)$, $(i,j) \in E$, and all negated atomic formulas $\neg(x_i = x_j)$, $(i,j) \notin E$. Any n-tuple of any infinite structure satisfies exactly one formula of the form $\varepsilon_E(x_1, \ldots, x_n)$, namely the one which expresses which of its coordinates are equal. For example, if $a \neq b$, then the triple (a, a, b) whose first two coordinates are equal and distinct from the third is associated to the equivalence relation E on coordinate set $\{1, 2, 3\}$ that partitions this set into E-classes $\{1, 2\}$ and $\{3\}$, the corresponding formula is

$$\varepsilon_E(x_1, x_2, x_3) : \quad (x_1 = x_2) \wedge \neg(x_1 = x_3) \wedge \neg(x_2 = x_3).$$

Each complete type contains some formula of the form ε_E, no complete type contains more than one ε_E, but it is possible that different complete types contain the same formula ε_E. (That is, it is possible for $\bar{a}$ and $\bar{b}$ to have different complete types with respect to the language L even if they have the same type with respect to the language of equality.)

Altogether, this shows that $t(n)$ is at least as large as the number of equivalence relations on an n-element set. (This number is called the n-th *Bell number*, and is sometimes written B_n.) On the other hand, an infinite pure M set has no more than B_n complete n-types. [To verify this we must argue that no two tuples with different complete types contain the same ε_E. Suppose to the contrary that $\bar{a}, \bar{b} \in M^n$ satisfy the same ε_E but $\mathrm{tp}^{\mathcal{M}}(\bar{a}) \neq \mathrm{tp}^{\mathcal{M}}(\bar{b})$. The partial mapping $a_i \mapsto b_i$, $1 \leq i \leq n$, can be extended to a permutation of the pure set M. A permutation of a pure set M is an automorphism of the structure $\langle M; \emptyset \rangle$, which must preserve types, hence $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{M}}(\bar{b})$. This contradicts the assumption that $\mathrm{tp}^{\mathcal{M}}(\bar{a}) \neq \mathrm{tp}^{\mathcal{M}}(\bar{b})$.] All in all, this shows that $|S_n^{\mathcal{M}}(\emptyset)| \geq B_n$ for any infinite structure and that equality holds when $\mathcal{M}$ is an infinite pure set. Hence the function t has been determined, it is $t(n) = B_n$. An infinite pure set $\mathcal{M}$ has exactly $t(n)$ complete n-types for each n, and the map defined by $p \mapsto E$ iff $\varepsilon_E \in p$ is a bijection from $S_n^{\mathcal{M}}(\emptyset)$ to the set of equivalence relations on $\{1, \ldots, n\}$. (Write p_E for p if $\varepsilon_E \in p$.)

Now we classify the structures $\mathcal{M} = \langle M; \mathcal{R}, \mathcal{F}, \mathcal{C} \rangle$ with $|S_n^{\mathcal{M}}(\emptyset)| = B_n$. It follows from the previous paragraph that each complete n-type is isolated by some complete formula ε_E. We want to understand the meaning of all formulas. That is, we want to say that, because a typical clopen set $O_{\varphi(\bar{x})}$ is determined by the points p_E that it contains, there should be some sense for which the meaning of the corresponding formula $\varphi(\bar{x})$ is determined by the corresponding complete formulas ε_E for $p_E \in O_{\varphi(\bar{x})}$. We establish such a fact now.

We have $\mathcal{M} \models \forall \bar{x}(\varepsilon_E(\bar{x}) \to \varphi(\bar{x}))$ whenever $\varphi(\bar{x}) \in p_E$, so

$$\mathcal{M} \models \forall \bar{x} \left(\left(\bigvee_{\varphi(\bar{x}) \in p_E} \varepsilon_E(\bar{x}) \right) \to \varphi(\bar{x}) \right).$$

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We can replace $\to$ with $\leftrightarrow$ in this displayed line for the following reason. If

$$\mathcal{M} \not\models \forall \bar{x} \left(\varphi(\bar{x}) \to \left(\bigvee_{\varphi(\bar{x}) \in p_E} \varepsilon_E(\bar{x}) \right) \right),$$

then

$$\mathcal{M} \models \exists \bar{x} \left(\varphi(\bar{x}) \wedge \neg \left(\bigvee_{\varphi(\bar{x}) \in p_E} \varepsilon_E(\bar{x}) \right) \right),$$

which implies that $\mathcal{M}$ has an n-tuple $\bar{a}$ such that $\operatorname{tp}(\bar{a})$ contains $\varphi(\bar{x})$, but does not contain any $\varepsilon_E(\bar{x})$ that strengthens $\varphi(\bar{x})$. This would force $\operatorname{tp}(\bar{a})$ to contain no $\varepsilon_E(\bar{x})$ at all, which is impossible since every tuple satisfies some formula $\varepsilon_E(\bar{x})$ that expresses the equalities among coordinates.

The previous paragraph shows that, when $|S_n^{\mathcal{M}}(\emptyset)| = B_n$, any formula $\varphi(\bar{x})$ is equivalent modulo $\operatorname{Th}(\mathcal{M})$ to a formula $\bigvee_{\varphi(\bar{x}) \in p_E} \varepsilon_E(\bar{x})$ in the language of equality.

Conversely, if $\mathcal{M}$ is such that any formula $\varphi(\bar{x})$ is equivalent modulo $\operatorname{Th}(\mathcal{M})$ to a formula expressed in the language of equality, then $\mathcal{M}$ has the same types as the underlying pure set, hence $|S_n^{\mathcal{M}}(\emptyset)| = B_n$.

We have found the function $t(n)$ and found that an infinite pure set has exactly $t(n)$ complete n-types for each n. We have also fully characterized the infinite structures $\mathcal{M} = \langle M; \mathcal{R}, \mathcal{F}, \mathcal{C} \rangle$ where $|S_n^{\mathcal{M}}(\emptyset)| = B_n$, namely they are the structures where every formula is equivalent modulo $\operatorname{Th}(\mathcal{M})$ to a formula expressed in the language of equality. Thus, our work is done, but let's briefly explore the implications of this conclusion.

First, $\operatorname{Aut}(\mathcal{M})$ must the full symmetric group on M. (This is another characterization of those structures $\mathcal{M}$ where $|S_n^{\mathcal{M}}(\emptyset)| = B_n$.) The fact that $\operatorname{Aut}(\mathcal{M})$ is the symmetric group on M implies, in particular, that $\mathcal{M}$ has no constant symbols. Next, here is an example of a nontrivial ternary relation expressible in the language of equality:

$$(a, b, c) \in R^{\mathcal{M}} \quad \text{iff} \quad (a = b \neq c) \vee (a \neq b = c).$$

Here is an example of a nontrivial ternary function expressible in the language of equality:

$$F^{\mathcal{M}}(a, b, c) = \begin{cases} c & \text{if } a = b \\ a & \text{else.} \end{cases}$$

These examples suggest what a typical relation or function of $\mathcal{M}$ might look like. (More specifically, if R is m-ary, then $R^{\mathcal{M}}$ is a union of $\operatorname{Sym}(M)$-orbits of M^m, while if F is n-ary, then $F^{\mathcal{M}}$ must be a function for which the relation $F^{\mathcal{M}}(x_1, \ldots, x_n) = x_{n+1}$ is a union of $\operatorname{Sym}(M)$-orbits of M^{n+1}.)