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10. In this problem a graph is a structure $G = \langle V; E \rangle$ where $E$ is a binary, irreflexive, symmetric relation on $V$.

(a) Classify the finite disconnected graphs $G$ such that $\mathrm{Th}(G)$ has quantifier elimination.

(b) Classify the finite graphs $G$ with disconnected complement such that $\mathrm{Th}(G)$ has quantifier elimination.

(c) It is known that every finite graph $G$ where $\mathrm{Th}(G)$ has quantifier elimination is of the type in part (a) or part (b) or else is one of the following two graphs: (i) the 5-element cycle, and (ii) the line graph of $K_{3,3}$. Draw pictures of these two graphs. (Include the definitions of $K_{3,3}$ and line graph in your solution.)

*Proof (a).* In this proof we will use the result of modthhw3p4:

**Theorem 1.** *A structure is ultrahomogeneous if every isomorphism between finitely generated substructures extends to an automorphism. If $A$ is a finite $L$-structure, then $Th(A)$ has quantifier elimination if and only if $A$ is ultrahomogenous.*

**Claim:** Let $G$ be a finite disconnected graph. $G$ is ultrahomogeneous if and only if $G$ is a disjoint union of isomorphic copies of $K_n$, where $K_n$ is the complete graph on $n$ vertices.

Let $G = \langle V; E \rangle$ be a finite, disconnected, ultrahomogeneous graph.

Choose any two vertices $u, v \in V$. The singleton substructures they generate are isomorphic, so by the ultrahomogeneity of $G$ there is an automorphism of $G$ that maps $u$ to $v$. This establishes that $\mathrm{Aut}(G)$ acts transitively on $V$, and one consequence of this is that all vertices of $G$ have the same degree.

We have assumed that $G$ is not connected, so it is possible to choose $u, v \in V$ from different components. Now choose distinct $w, z \in V$ that are not adjacent to each other (i.e. $E(w, z)$ fails). The doubleton substructures $\langle \{u, v\}; E \rangle$ and $\langle \{w, z\}; E \rangle$ are both subgraphs that are discrete (meaning: no edges), hence they are isomorphic. By ultrahomogeneity there must be an automorphism of $G$ mapping $\{u, v\}$ to $\{w, z\}$. But automorphisms preserve connected components, and $u$ and $v$ are in different components, so we conclude that $w$ and $z$ must lie in different components. The conclusion to draw is that if $w$ is not connected by a single edge to $z$, then $w$ cannot be connected to $z$ by a path of any length. Equivalently, the connected components of $G$ are complete. This establishes that $G$ is the disjoint union of complete components $G = K_{n_1} \sqcup \cdots \sqcup K_{n_r}$. But we established in the preceding paragraph that all vertices of $G$ have the same degree, so $n_1 = \cdots = n_k$; let this common value be called $n$. This completes the proof of the forward direction of the Claim.

Now we argue the reverse direction. Assume that $G$ is a finite disjoint union of copies of $K_n$ for some $n$. Our goal is to prove that $G$ is ultrahomogeneous. Let $f : M \to N$ be an isomorphism between substructures of $G$. Our goal is to show that $f$ extends to an automorphism of $G$. It will suffice, by finiteness, to show that if $|M| < |G|$ then it is possible to extend $f$ to an isomorphism $f^* : M^* \to N^*$ where $M^*, N^*$ each have one more vertex than $M, N$ respectively. (We will write $V_G, V_M, V_{M^*}$ etc to indicate the vertex set of a given substructure of $G$.)

Write $M$ as a disjoint union of its components $M = M_1 \sqcup \cdots \sqcup M_k$ and let $N = f(M_1) \sqcup \cdots \sqcup f(M_k) = N_1 \sqcup \cdots \sqcup N_k$.

**Case 1.** $G$ has more components than $M$.

In this case, there exists $u \in V_G$ that belongs to none of the components of $G$ containing any $M_i$. Now $N$ has the same number of components as $M$, so there must also exist an element $v \in V_G$ that belongs to none of the components of $G$ containing any $N_i$. Define $V_{M^*} = V_M \cup \{u\}$ and $V_{N^*} = V_N \cup \{v\}$. We have enlarged both $M$ and $N$ by adding a singleton connected component to each. We can extend $f$ to $f^* : M^* \to N^*$ by $f^*|_M = f$ and $f^*(u) = v$. To check that $f^*$ is an isomorphism, we must verify that it maps adjacent pairs to adjacent pairs, and nonadjacent pairs to nonadjacent pairs. This is easy to see, because the vertex added to each subgraph is not adjacent to any of the earlier vertices.

**Case 2.** Some component of $M$ is properly contained in some component of $G$.

Suppose that some component $M_i$ of $M$ is properly contained in some $G$-component. This will be true of $M_i$ if and only if $|V_{M_i}| < n$. Since $|V_{M_i}| = |V_{N_i}|$, if this is true for $M_i$ it will also be true for $N_i$.

Choose elements $u, v \in V_G$ such that $u \notin V_{M_i}$ and $v \notin V_{N_i}$, but $u$ belongs to the $G$-component of $M_i$ and $v$ belongs to the $G$-component of $N_i$. Define $V_{M^*} = V_M \cup \{u\}$ and $V_{N^*} = V_N \cup \{v\}$. We have enlarged both $M$ and $N$ by expanding their $i$th components to larger complete subgraphs. We can extend $f$ to $f^* : M^* \to N^*$ by $f^*|_M = f$ and $f^*(u) = v$. To check that $f^*$ is an isomorphism, we must verify that it maps adjacent pairs to adjacent pairs, and nonadjacent pairs to nonadjacent pairs. For this it is enough to observe that $u$ is distinct from every vertex in $V_M$, it is adjacent to every vertex in $V_{M_i}$, and it is not adjacent to any other vertex in $V_M$, while $v$ has the same properties with respect to $N$.

The two cases cover all cases where $M$ and $N$ are proper subgraphs of $G$. Since any isomorphism between proper subgraphs can always be extended by one point, we can continue these extensions until an automorphism of $G$ is obtained.

*Proof (b).* In this part we argue that if $G$ is a graph with the property that $\mathrm{Th}(G)$ has quantifier elimination, then the complement $G'$ also has the property that $\mathrm{Th}(G')$ has quantifier elimination. Here if $G = \langle V; E \rangle$ is the graph with vertex set $V$ and irreflexive edge relation $E(x, y)$, then $G'$ is defined to be the graph $\langle V; E' \rangle$ with the same vertex set and the irreflexive edge relation $E'(x, y)$ defined by the formula

$$(x \neq y) \wedge \neg E(x, y) \tag{$\dagger$}$$

That is, a nonloop edge belongs to $G$ iff it does not belong to $G'$.

We derive the result from a more general statement:

**Theorem 2.** *Let $L$ and $L'$ be relational languages. Let $\mathbb{A}$ be an $L$-structure and let $\mathbb{A}'$ be an $L'$-structure. Assume that*

1. *$\mathbb{A}$ and $\mathbb{A}'$ share the same underlying set, $A$.*

2. *Each fundamental relation $R(x)$ of $L$ has a quantifier-free $L'$-definition. That is, there exists a quantifier-free formula $\rho'(x)$ of $\mathbb{A}'$ such that $R[\mathbb{A}] = \rho'[\mathbb{A}']$. (The same tuples from $A^n$ are in $R$-relation as those in $\rho'$-relation.)*

3. *Each fundamental relation $R'(x)$ of $L'$ has a q.f. $L$-definition.*

*Then $Th(\mathbb{A})$ has quantifier elimination iff $Th(\mathbb{A}')$ has quantifier elimination.*

Roughly this says that if $\mathbb{A}$ and $\mathbb{A}'$ are "the same structure" up to a change of language that respects the property of a formula of being quantifier-free, then $\mathrm{Th}(\mathbb{A})$ has quantifier elimination iff $\mathrm{Th}(\mathbb{A}')$ has quantifier elimination.

Without writing out the proof of this theorem, we simply point out the following

1. The correspondences $R \mapsto \rho'$ and $R' \mapsto \rho$ can be extended to correspondences between arbitrary $L$-formulas to arbitrary $L'$-formulas which maps any q.f.-$L$-formula $\alpha(x)$ to a q.f.-$L'$-formula $\alpha'(x)$ in a way that preserves interpretations ($\alpha[\mathbb{A}] = \alpha'[\mathbb{A}']$).

2. Assume that $\mathrm{Th}(\mathbb{A})$ has q.e., and let's argue that $\mathrm{Th}(\mathbb{A}')$ also has it. Choose any $L'$-formula $\varphi'(x)$. Let $\varphi(x)$ be the corresponding $L$-formula. Let $\alpha(x)$ be the q.f.-$L$-formula equivalent to $\varphi(x)$. Let $\alpha'(x)$ be the q.f.-$L'$-formula corresponding to $\alpha(x)$. Then $\alpha'(x)$ will be a q.f.-$L'$-formula equivalent to $\varphi'(x)$. Hence $\mathrm{Th}(\mathbb{A}')$ has q.e.

So, to complete the argument for (b), it suffices to show that the edge relation $E'$ has a q.f.-definition in terms of $E$ (and vice versa). That was already done above at (†). (For the "vice versa" part, interchange the roles of $E$ and $E'$.)

*(c).*

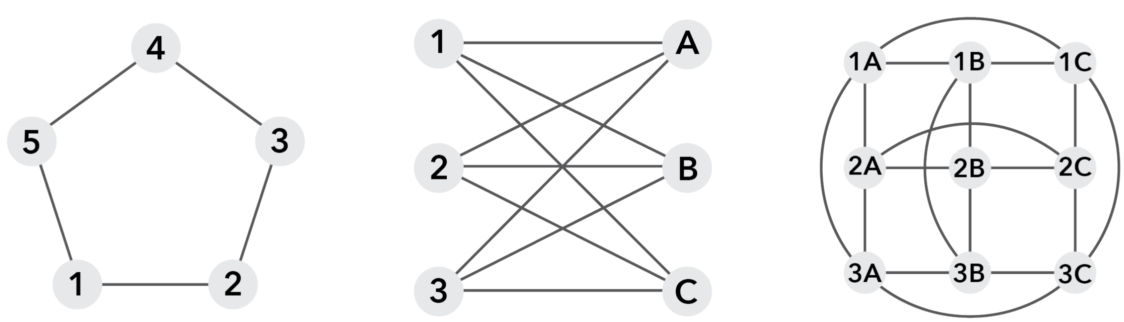


Figure 1: the 5-element cycle, $K_{3,3}$, and $L(K_{3,3})$

A complete bipartite graph is a graph whose vertices can be partitioned into two subsets $V_1$ and $V_2$ such that each vertex in $V_1$ connects to every vertex in $V_2$ and that there are no edges between elements within each partition.

Given a graph $G$, the line graph $L(G)$ is a graph such that each edge in $G$ is interpreted as a single vertex in $L(G)$. Edges in the graph $G$ which are adjacent appear as vertices in the graph $L(G)$ which are adjacent.

□