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3. Suppose that L is a language and L' is an expansion of L by some set C of additional constant symbols. Suppose that T is an L-theory that has quantifier elimination and that $T' \supseteq T$ is an L'-theory extending T. Show that T' has q.e.

Proof. Let L be a language, and let L' be an expansion of L by some set C of additional constant symbols. Suppose that T is an L-theory that has quantifier elimination, and that $T' \supseteq T$ is an L'-theory extending T. Consider an arbitrary L'-formula, written $\phi(\mathbf{x}, \mathbf{c})$ where $\phi(\mathbf{x}, \mathbf{y})$ is an L-formula and $\mathbf{c}$ is a tuple of constants from C. Since T has quantifier elimination, there is a quantifier-free L-formula $\alpha(\mathbf{x}, \mathbf{y})$ so that

$$T \models (\forall(\mathbf{x}, \mathbf{y}))(\phi(\mathbf{x}, \mathbf{y}) \leftrightarrow \alpha(\mathbf{x}, \mathbf{y})).$$

Since $T' \supseteq T$,

$$T' \models (\forall(\mathbf{x}, \mathbf{y}))(\phi(\mathbf{x}, \mathbf{y}) \leftrightarrow \alpha(\mathbf{x}, \mathbf{y})).$$

Then by letting $\mathbf{y} = \mathbf{c}$,

$$T' \models (\forall \mathbf{x})(\phi(\mathbf{x}, \mathbf{c}) \leftrightarrow \alpha(\mathbf{x}, \mathbf{c})),$$

showing that T' has quantifier elimination. $\square$

The theory of dense linear order without endpoints has q.e. (you may assume this). Show that any theory of dense linear order with some additional constant symbols is complete iff the theory completely decides how the order relation restricts to the interpretations of the constant symbols.

Proof. Let T be the theory of dense linear order with some additional constant symbols.
($\Rightarrow$) Suppose T is complete. Then, for any $a, b \in C$, one of $a < b$ or $\neg(a < b)$ is in T.
($\Leftarrow$) Suppose T completely decides how the order relation restricts to the interpretations of the constant symbols. Since the theory of dense linear order (DLO) has quantifier elimination, by the previous result we have that T has quantifier elimination. Therefore, to show that T is complete is to show that T decides all quantifier free sentences. By disjunctive normal form it is enough to look at atomic sentences (and their negations), which have the form $a = b$, or $a < b$ where $a, b \in C$. But, $a < b$ and $a = b$ are decided by T by assumption, in addition to $\neg(a < b) \equiv (b < a) \vee (b = a)$ and $\neg(a = b) \equiv (a < b) \vee (b < a)$. Therefore T is complete. $\square$