MATH 6000 - Assignment 2

Christopher Eblen, Trevor Manders, Andrew Stocker

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8. Show that an ultraproduct of finite structures over the index set $I = \omega$ is either finite or of size $2^{\aleph_0}$.

Proof. Let L be a language, let $\{\mathbb{A}_i\}_{i\in\omega}$ be a collection of finite L-structures, and let $\mathcal{U}$ be an ultrafilter on ω

Case I: Suppose there is $N \in \mathbf{N}$ and $U \in \mathcal{U}$ such that $|\mathbb{A}_i| \leq N$ for all $i \in U$. Let σ_N be an L-sentence such that $\mathbb{A} \models \sigma_N$ if and only if $|\mathbb{A}| \leq N$, for any L-structure $\mathbb{A}$. Then, if $i \in U$, we have $|\mathbb{A}_i| \leq N$ and hence $\mathbb{A}_i \models \sigma_N$. Therefore, $[\![\sigma_N]\!] = \{i \in \omega : \mathbb{A}_i \models \sigma_N\} \supseteq U$, and so $[\![\sigma_N]\!] \in \mathcal{U}$ since $\mathcal{U}$ is a filter. By Łos's Theorem, we have that $\prod_{\mathcal{U}} \mathbb{A}_i \models \sigma_N$ and hence $|\prod_{\mathcal{U}} \mathbb{A}_i| \leq N$.

Case II: Now suppose that for every $U \in \mathcal{U}$, the sequence $\{|\mathbb{A}_i|\}_{i\in U}$ is unbounded. Partition ω into sets

$$I_k := \{i \in \omega : 2^k \leq |A_i| < 2^{k+1}\}$$

for each $k \in \mathbf{N}$. Note that, for any $k \in \mathbf{N}$, the sequence $\{|\mathbb{A}_i|\}_{i\in I_k}$ is bounded by 2^{k+1}, so no I_k is in $\mathcal{U}$. For each $k \in \mathbf{N}$ and each $\mathbb{A}_i$ such that $i \in I_k$, label 2^k elements of $\mathbb{A}_i$ by $a_{i,\sigma}$, where $\sigma \in \{0,1\}^k$ is a binary sequence of length k (of which there are exactly 2^k). For a binary sequence $p \in \{0,1\}^\omega$, denote by p_k the binary sequence of length k obtained from p by taking only the first k terms. Define a map

$$\begin{aligned} f : \{0,1\}^\omega &\to \prod\nolimits_{\mathcal{U}} \mathbb{A}_i \\ p &\mapsto [t(p)]_{\vartheta_{\mathcal{U}}}, \end{aligned}$$

where $t(p) = (a_{i,p_k})_{i\in\omega}$ (the i^{th} entry of the tuple $t(p)$ is the unique element a_{i,p_k}, where k is the index of the set I_k in which i is found). Suppose $p \neq q$ are two elements of $\{0,1\}^\omega$. Then there is some $N \in \mathbf{N}$ such that $p_N = q_N$ but $p_{N+j} \neq q_{N+j}$ for any $n > 0$; we take $N = 0$ to mean that the first term of p differs from that of q. To see $[t(p)]_{\vartheta_{\mathcal{U}}} \neq [t(q)]_{\vartheta_{\mathcal{U}}}$, we must show that $[\![t(p) = t(q)]\!] \notin \mathcal{U}$. If $i \in [\![t(p) = t(q)]\!]$, then $i \in I_k$ for some k and $a_{i,p_k} = a_{i,q_k}$.

But then p and q have the same first k terms, and since the largest possible such k is N, we have $k \leq N$ and hence $2^k \leq |\mathbb{A}_i| < 2^{k+1} \leq 2^{N+1}$. Therefore, the sequence $\{|\mathbb{A}_i|\}_{i \in [\![t(p)=t(q)]\!]}$ is bounded by 2^{N+1} , so that $[\![t(p) = t(q)]\!] \notin \mathcal{U}$. This shows that f is injective. Since $\{0,1\}^\omega$ has cardinality $2^{\aleph_0}$, we have $|\prod_{\mathcal{U}} \mathbb{A}_i| \geq 2^{\aleph_0}$. But $|\prod_{\mathcal{U}} \mathbb{A}_i| \leq 2^{\aleph_0}$ because the $\mathbb{A}_i$ are finite, so that $|\prod_{\mathcal{U}} \mathbb{A}_i| = 2^{\aleph_0}$.

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