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In the Language of one unary predicate $P(x)$ the class of structures $\langle A; P(x)\rangle$ where $|P[A]| = |A - P[A]|$ is not elementary.

Proof. let $p(x) = \begin{cases} true & x \in \mathbb{Z} \\ false & else \end{cases}$ and let $A = \mathbb{R}$, this does not model this theory because $\mathbb{Z}$ and $\mathbb{R} - \mathbb{Z}$ have different cardinalities.

By the Löwenheim-Skolem theorem there exists an countable elementary substructure.

In this substructure P(x) remains the same however A is a countable subset of $\mathbb{R}$, and $P[X]$ is a countable subset of $\mathbb{Z}$. Because this is an elementary substructure we can conclude that $P[A]$ and $A - P[A]$ are countably infinite by the sentences
$\exists x_1, \ldots, \exists x_n((\bigwedge_{i<j}(x_i \neq x_j)) \wedge (\bigwedge_{i=1}^{n}(P(x_i) = true)))$ and
$\exists x_1, \ldots, \exists x_n((\bigwedge_{i<j}(x_i \neq x_j)) \wedge (\bigwedge_{i=1}^{n}(P(x_i) = false)))$.
Therefore $A - P[A]$ and $P[A]$ have the same cardinality.
The substructure when A is countable is in the class, while when $A = \mathbb{R}$ the structure is not, so this class of structures is not an elementary class. □

In the Language of one unary predicate $P(x)$, and one function F, where F is a bijection between $P[A]$ and $A - P[A]$ the class of structures $|P[A]| = |A - P[A]|$ is elementary.
Define F(x) such that if P(x) is true then P(F(x)) is false and if P(F(x)) is true then P(x) is false,

Proof. To show this is an elementary class we will show that 1, every model of this theory is in the class of sets that can be partitioned into 2 equal cardinality subsets by a unary predicate. and 2, every set that can be partitioned into 2 equal cardinality subsets is a model of this theory.

1. If A is a model of this theory then A can be partitioned into 2 equal cardinality subsets.

2. Let A be a set that can be partitioned into 2 equal cardinality subsets, B and C then there exists a bijection between B and C, and let $p(x) = \begin{cases} true & x \in B \\ false & else \end{cases}$

□

Therefore the class of structures $|P[A]| = |A - P[A]|$ is pseudo-elementary but not elementary.

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Proof. Pseudo-elementary classes are closed under ultra-products.
Let A_i, $i \in I$ be pseudo-elementary classes in the signature σ. Expand the signature to σ' so that the class is an elementary class, because elementary classes are closed under ultra-products $\Pi_I A'_i$ is a model of the theory of the richer signature, so it will also model the theory of the original signature. $\square$