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11.Look up the definitions of Σ_1^1 and Π_1^1 sentences if necessary, and include the definitions in your solution to the following problems.

a Show that ultraproducts preserve Σ_1^1 sentences.

b Give an example of a Π_1^1 sentence not preserved by ultraproducts.

$\Sigma_0^1 = \Pi_0^1$ denote the class of formulae in second order logic with no quantification over sets. (That is all second order variables are free). We define Σ_1^1 to be the class of formulae that are of the form $(\exists X_1)(\exists X_2), \ldots (\exists X_k)\psi$, where ψ is a formula of class Π_0^1. Similarly Π_1^1 are the class of formulae of the form $(\forall X_1)(\forall X_2) \ldots (\forall X_k)\psi$ where ψ is a formula of class Σ_0^1. Where $X_1 \ldots X_k$ quantify over functions, relations, constants and sets of domain elements.

Proof.

a Let us consider a sentence $\phi = (\exists R_1)(\exists R_2) \ldots (\exists R_k)\varphi(R_1, R_2 \ldots R_k)$, where ψ is a formula with bound first order variables and R_i are n_i-ary relations, (we can consider relations as they can simulate functions, constants and sets of domain elements, in which case we use a unary function to denote presence or absence) .
We assume that each $\mathbb{A}_i$ satisfies ϕ, so it is possible to choose relations $\rho_{1,i}, \ldots, \rho_{k,i}$ on $\mathbb{A}_i$ such that $\mathbb{A}_i \models \varphi(\rho_{1,i}, \ldots, \rho_{k,i})$. Now expand the language to include new relations symbols of the appropriate arity and define an expansion $\mathbb{B}_i$ of $\mathbb{A}_i$ for which $R_j^{\mathbb{B}_i} = \rho_{j,i}$. The structure $\mathbb{B}_i$ now satisfies $\varphi(R_1, \ldots, R_k)$. If we consider $\prod_U \mathbb{B}_i$ that there must be relations $\rho_{1,\mathcal{U}}, \ldots \rho_{k,\mathcal{U}}$ which satisfy $\varphi(\rho_{1,\mathcal{U}}, \ldots \rho_{k,\mathcal{U}})$. Therefore $\prod_U \mathbb{B}_i \models \varphi(R_1, \ldots, R_k)$. Therefore $\prod_U \mathbb{A}_i \models (\exists R_1) \cdots (\exists R_k)(\varphi(R_1, \ldots, R_k)$. Thus we find that the ultraproduct also satisfies the sentence, and as such ultraproducts must preserve Σ_1^1 sentences.

b We can write first order sentences denoting a given function f is injective $(\phi(f))$ and similarly that it is surjective $(\psi(f))$. We now consider models $\mathbb{A}_i$ such that A_i has i elements. Now we consider the Π_1^1 sentence $\phi' = \forall f(\phi(f) \to \psi(f))$, which states that all functions which are injective must be surjective. From the definition of Dedekind finiteness we know this must be satisfied by all functions with a finite domain. Therefore ϕ' must be true in every finite set A_i .We now consider the ultraproduct of the $\mathbb{A}_i$'s over a non-principal ultrafilter with an index set of size ω, and see that it's size is infinite, and hence will fail the sentence $(\forall f)(\phi(f) \to \psi(f))$. This shows that it is possible for an ultraproduct to fail a Π_1^1 sentence even when each factor of the ultraproduct satisfies the sentence.