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1. Let $\mathcal{L}$ be any countable language. Show that for any infinite cardinal κ there are at most 2^κ nonisomorphic $\mathcal{L}$-structures of cardinality κ.

Proof. Let $\mathcal{L}$ be any countable language. Let the associated signature be $(\mathcal{F}, \mathcal{R}, ar)$. Since $|\mathcal{F}|$ and $|\mathcal{R}|$ are countable, let's say $|\mathcal{F}|+|\mathcal{R}| = \aleph_0$. Every $\mathcal{L}$-structure of size κ is isomorphic to an $\mathcal{L}$-structure with domain κ, so to count isomorphism types, we can instead count isomorphism types of structures $\mathbb{A}$ with domain $A = \kappa$. Since we only want to know that there are at most 2^κ isomorphism types, we just need to find an upper bound on these isomorphism types. We do this by finding an upper bound on all possible structures. Structures differ by the tables associated with their relations, so we will consider all possible tables and count them. For each function symbol, the arity changes the number of tables possible. Consider a function of arity 1. Then the number of possible tables is κ^κ since for each of the κ-many entries in the table, we have κ-many choices. However, we have the following inequality: $2 \leq \kappa \leq 2^\kappa$. The second part of the inequality is by Cantor's Theorem. Exponentiating,

$$2^\kappa \leq \kappa^\kappa \leq (2^\kappa)^\kappa$$

Then,

$$2^\kappa \leq \kappa^\kappa \leq 2^{\kappa \cdot \kappa}$$

Next,

$$2^\kappa \leq \kappa^\kappa \leq 2^\kappa$$

So, $\kappa^\kappa = 2^\kappa$ by the above inequality. Then, we have 2^κ-many possible tables of unary functions. Then, a function of arbitrary arity $n \in \omega$ similarly has $\kappa^{(\kappa^n)} = \kappa^\kappa = 2^\kappa$-many tables. So functions of arity $ar \in \omega$ must have 2^κ-many tables since the number of tables is bound by the tables for arity 1 and arity n. Next, we consider relations, which are bound in the same way as functions. A 1-ary relation has 2^κ-many combinations in its table since for each of the κ-many entries we can have either true or false. An n-ary relation has $2^{(\kappa^n)} = 2^\kappa$-many tables, so all relations with arity $ar \in \omega$ have 2^κ tables. Finally, for constants. There are κ-many choices for each constant and β-many possible constant symbols, where β is a countable cardinal. Then, an upper bound on the number of possible choices for constants is $\kappa \cdot \kappa \cdot \kappa \cdots \kappa$ (β-many times). This is then $\leq \kappa^\beta \leq \kappa^\kappa = 2^\kappa$-many constant interpretations. Thus, an upper bound on our total number of possible structures (including isomorphic structures) is $2^\kappa \cdot 2^\kappa \cdot 2^\kappa \cdot \cdots \cdot 2^\kappa$ ($\aleph_0 + 1 = \aleph_0$-many times). This product is $\leq (2^\kappa)^{\aleph_0} = 2^\kappa$. Hence, there are at most 2^κ non-isomorphic $\mathcal{L}$-structures of cardinality κ.

$\square$