14. Show that $\exists x_1 \forall x_2 P(x_1, x_2) \to \forall x_2 \exists x_1 P(x_1, x_2)$ is logically valid, but not a tautology.

Proof. First we will show that that the above sentence is logically valid. Consider each of the sentences $\varphi_1 = \exists x_1 \forall x_2 P(x_1, x_2)$ and $\varphi_2 = \forall x_2 \exists x_1 P(x_1, x_2)$. We then want to evaluate the truth of $\varphi_1 \to \varphi_2$. If φ_2 evaluates as false, that is, for some particular x_2' there is no x_1 such that $P(x_1, x_2')$ is true, then it must also be the case that φ_1 evaluates as false since for any given x_1, $P(x_1, x_2')$ cannot be true. Note that φ_2 could evaluate to true while φ_1 evaluates to false, for example in $\mathbb{R}$ with $P(x_1, x_2) = x_1 \leq x_2$, or in the case that the model is empty. However since $\neg \varphi_2 \to \neg \varphi_1$ in any valuation we can deduce that $\varphi_1 \to \varphi_2$ is logically valid.

However, this sentence is not a tautology. To see this, consider $\exists x_1 \forall x_2 P(x_1, x_2) \to \forall x_2 \exists x_1 P(x_1, x_2)$ as a tree. At the root is $\to$, but then the rest of the tree comprises two non-equivalent subtrees which each begin with quantifiers. Hence this statement must have come from substituting sentences for P and Q in the sentence $P \to Q$, or from substituting the whole thing for P in the sentence P, neither of which are propositional tautologies. $\square$