9. (a) Let $\mathcal{U}$ be an ultrafilter on a set I and let $\{\mathbb{A}_i | i \in I\}$ be a set of L-structures. Show that $\prod_{\mathcal{U}} \mathrm{Aut}(\mathbb{A}_i)$ is embeddable in $\mathrm{Aut}(\prod_{\mathcal{U}} \mathbb{A}_i)$. (Elements of $\prod_{\mathcal{U}} \mathrm{Aut}(\mathbb{A}_i)$ are called internal automorphisms of $\prod_{\mathcal{U}} \mathbb{A}_i$ while other automorphisms are called external.)

Proof: Let $[f]$ denote the equivalence class of $f \in \prod_{i\in I} \mathrm{Aut}(\mathbb{A}_i)$, i.e. $[f] = \{g \in \prod_{i\in I} \mathrm{Aut}(\mathbb{A}_i) | [\![f = g]\!] \in \mathcal{U}\}$.

Let

$$\phi : \prod_{\mathcal{U}} \mathrm{Aut}(\mathbb{A}_i) \to \mathrm{Aut}(\prod_{i\in I} \mathbb{A}_i)$$

$$[f] \mapsto f.$$

Let

$$\pi : \mathrm{Aut}(\prod_{i\in I} \mathbb{A}_i) \to \mathrm{Aut}(\prod_{\mathcal{U}} \mathbb{A}_i)$$

$$f \mapsto f/\theta_{\mathcal{U}}.$$

The map $\pi \circ \phi$ is well-defined since $[f] = [g]$ implies $[\![f = g]\!] \in \mathcal{U}$. ϕ is an injection since $f/\theta_{\mathcal{U}} = g/\theta_{\mathcal{U}}$ implies $f(i) = g(i)$ $\mathcal{U}$-almost everywhere. Hence $[f] = [g]$.

(b) Give an explicit example of an external automorphism.

Let $A_i = \{0, 1\}$ for all $i \in \omega$. Let $\mathbb{A}_i = \langle A_i; \{f_j | j \in \omega\} \cup \{s_i\}\rangle$ where f_j is the unary identity function on A_i if $j \neq i$ and f_i is the constant zero function if $j = i$, and s_i is the unary function on A_i mapping 1 to 0 and 0 to 1. Then $\mathrm{Aut}(\mathbb{A}_i)$ is trivial for all i as follows. Let $\alpha \in \mathrm{Aut}(\mathbb{A}_i)$. Then $\alpha(0) = \alpha(f_i(0)) = f_i(\alpha(0)) = 0$, and so $\alpha(1) = 1$ since α is a bijection.

Let $\mathcal{U}$ be a non-principal ultrafilter on ω. Then $|\prod_{i\in\omega} \mathrm{Aut}(\mathbb{A}_i)| = 1$ and so $|\prod_{\mathcal{U}} \mathrm{Aut}(\mathbb{A}_i)| = 1$, i.e. the only internal automorphism is the identity.

Let $\mathbb{B} = \prod_{\mathcal{U}} \mathbb{A}_i$ Then $\mathrm{Aut}(\mathbb{B})$ contains a non-identity automorphism, namely $[(s_i)_{i\in\omega}] := s^{\mathbb{B}}$. First $s = (s_i)_{i\in\omega}$ is a bijection from $\prod_{i\in\omega} A_i$ to itself since $s \circ s$ is the identity on $\prod_{i\in\omega} A_i$. Hence $s^{\mathbb{B}}$ is a bijection from $\prod_{\mathcal{U}}$ to itself.

Now $s^{\mathbb{B}}$ is an automorphism as follows: Clearly $s^{\mathbb{B}}$ commutes with the identity function and with itself. Now $f_j^{\mathbb{B}}$ commutes with $s^{\mathbb{B}}$ for all $j \in \omega$ as follows. $(f_j)_{j\in\omega}$ is the identity in all coordinates except the j^{th} coordinate. Since $\mathcal{U}$ is a non-principal ultrafilter, $[\![(f_j)_{i\in\omega} = id]\!] \in \mathcal{U}$ where id is the identity function on $\prod_{i\in\omega} A_i$. Hence $f_j^{\mathbb{B}}$ is the identity function on $\mathbb{B}$ and so $s^{\mathbb{B}}$ commutes with $f_j^{\mathbb{B}}$ for each $j \in \omega$. Therefore $s^{\mathbb{B}}$ is an automorphism of $\mathbb{B}$.

Lastly, $[\![(s_i)_{i\in\omega} \neq id]\!] = \omega \in \mathcal{U}$. Hence in $\mathbb{B}$, $s^{\mathbb{B}}$ is not the identity. Therefore $s^{\mathbb{B}}$ is an external automorphism of $\mathbb{B}$.