Let Th_L be the lattice of all L-theories for some language L. Any atom in this lattice has a complement, any complement of an atom is a coatom (and vice-versa), but that there must exist at least one coatom that does not have a complement.

Proof.

Let L be a language. Let Th_L be the lattice of all L-theories. Note that the top element of this lattice is $\{falsity\}$ we call α and the bottom element is $\emptyset^{\perp\perp}$ we call β. First, we show that any atom in the lattice has a complement. Consider an arbitrary atom in this lattice T. Since T is a atom, it is axiomatized by a single sentence, say $T = \{\sigma\}^{\perp\perp}$. Then, consider the element of the lattice $\{\neg\sigma\}^{\perp\perp}$. To be a complement, we must have $\{\sigma\}^{\perp\perp} \vee \{\neg\sigma\}^{\perp\perp}$ be the top element, α and , $\{\sigma\}^{\perp\perp} \wedge \{\neg\sigma\}^{\perp\perp}$ be the bottom element β. For the first part, satisfying a sentence and its negation only happens when falsity is satisfied, so this join is clearly the top. For the second condition, we consider the sentences implied by $\{\sigma\}^{\perp\perp}$ and $\{\neg\sigma\}^{\perp\perp}$. Suppose that $\{\sigma\}^{\perp\perp} \wedge \{\neg\sigma\}^{\perp\perp}$ was not the bottom element. Then there is a sentence, τ, that is not logically validity such that $\sigma \models \tau$ and $\neg\sigma \models \tau$. Then we have, $\neg\tau \models \neg\sigma$ and $\neg\tau \models \neg\neg\sigma$. Hence, we have $\neg\tau \models falsity$. Thus, $truth \models \tau$. So, $\{\sigma\}^{\perp\perp}$ and $\{\neg\sigma\}^{\perp\perp}$ cannot imply the same sentence unless that sentence is logically valid. Thus, every atom has a complement.

Now, we show that every complement of an atom is a coatom. Assume that the complement of some atom is not a coatom. Call the atom A, the complement C, and some element B such that $C < B < \alpha$, such that $B = B \wedge (A \vee C)$. We know this B exists because if it did not, C would satisfy the conditions of being a coatom. But then by the distributive law $B = (B \wedge C) \vee (B \wedge A) = C \vee \beta = C$. Hence, C must have been a coatom or the distributive law would fail. Then every complement of an atom is a coatom. Similarly, we have that every complement of a coatom is an atom. Assume B is a coatom, C is the complement of B and A is some element such that $\alpha < A < C$, such that $A = A \vee (B \wedge C)$. We know this A exists because otherwise C would satisfy the conditions for being an atom. Then we have $A = (A \vee C) \wedge (A \vee B) = C \wedge \alpha = C$. Thus, A must have been C or the distributive law would fail. So every complement of a coatom is an atom.

Lastly, we show there must exist at least one coatom that does not have a complement. Assume not. Assume every coatom has a complement. In class, we showed that if a theory has a complement, then it is finitely axiomatizable. Thus every coatom must be finitely axiomatizable, since they are theories with complements. This means in Spec(L), the space of complete L-theories, every point is isolated. Spec(L) has a discrete topology then. Since Spec(L) is compact and discrete, it only has finitely many points. Then, we have that every coatom is finitely axiomatizable and there are only finitely many points in Spec(L), so there are only finitely many L-structures up to elementary equivalence. However, every language has a model of size n, for each n $\in \omega$. Then these cannot be elementarily equivalent, so we have reached a contradiction. Hence, there must be some coatom that does not have a complement. □