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6. An (nonprincipal) ultrafilter is uniform if all of its sets have the same size. Show that a regular ultrafilter is uniform.

Proof. Let $\mathcal{U}$ be a regular ultrafilter on a infinite set I. Then, there is a subset $\mathcal{E} \subset \mathcal{U}$ such that $|\mathcal{E}| = |I|$ and $\mathcal{E}_i = \{E \in \mathcal{E} : i \in E\}$ is finite for all $i \in I$.

To prove that $\mathcal{U}$ is uniform, we need to show every $U \in \mathcal{U}$ has the same size. Since $\mathcal{U}$ is an ultrafilter, $\emptyset \notin \mathcal{U}$ and $I \in \mathcal{U}$. Hence, if we can show $|I| = |U|$ for any $U \in \mathcal{U}$, $\mathcal{U}$ will be uniform. Fortunately, it is clear that $|I| \geq |U|$ (since $I \supset U$). So we just have to show $|I| \leq |U|$.

The core idea of this proof is to abuse some quantitative relation between $\mathcal{E}$ and $U \in \mathcal{U}$. So let U be an arbitrary set in $\mathcal{U}$ and E be an arbitrary set in $\mathcal{E}$. Since $\mathcal{E} \subset \mathcal{U}$, and so $E \in \mathcal{U}$, $E \cap U \neq \emptyset$ by the Finite Intersection Property. This means that $E \in \mathcal{E}_i$ for some $i \in U$. Thus,

$$\begin{aligned}\mathcal{E} &= \{E \in \mathcal{E}\} \\ &= \{E \in \mathcal{E} : E \in \mathcal{E}_i \text{ for some } i \in U\} \\ &= \bigcup_{i \in U} \mathcal{E}_i.\end{aligned}$$

Finally, by regularity of $\mathcal{U}$, we have

$$\begin{aligned}|I| &= |\mathcal{E}| \\ &= |\bigcup_{i \in U} \mathcal{E}_i| \\ &\leq \sum_{i \in U} |\mathcal{E}_i|.\end{aligned}$$

(Notice that U is infinite since otherwise, the sum above will be finite, and that implies, I is finite).

$$\begin{aligned}&\leq \sum_{i \in U} \omega \\ &= |U|\omega \\ &= |U|.\end{aligned}$$

Therefore, $|I| = |U|$ for any $U \in \mathcal{U}$. So, a regular ultrafilter is uniform. $\square$