MODEL THEORY HOMEWORK 2

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Problem (10)**.** Let $\mathbb{A}$ be a an L-structure. Then the following are equivalent:

(1) $\mathbb{A}$ is a model of the theory of finite L-structures.
(2) Every sentence true in $\mathbb{A}$ holds in some finite L-structure.
(3) $\mathbb{A}$ is elementarily equivalent to an ultraproduct of finite L-structures.

Proof. (1) $\Rightarrow$ (2). Let F be the theory of finite L-structures and suppose $\mathbb{A} \models F$. Suppose toward contradiction that σ is a sentence that is true in $\mathbb{A}$ but does not hold in *any* finite L-structure. Then $\neg\sigma$ holds in *every* finite L-structure; that is, $\neg\sigma \in F$, but then $\mathbb{A} \models \neg\sigma$, contradicting the supposition that $\mathbb{A} \models \sigma$.

(2) $\Rightarrow$ (3). Let T be the (complete) theory of $\mathbb{A}$, and let $I = \mathcal{P}_\omega(T)$ be the set of finite subsets of T. For each $X \in I$, consider the single sentence $\sigma_X = \bigwedge_{s \in X} s$ and let $\mathbb{A}_X$ be a finite L-structure where σ_X holds. Let $\mathcal{F} = \{\uparrow X : X \in I\}$ be the set of principal upper sets of I (as a poset under inclusion). $\mathcal{F}$ has the finite intersection property since, for any finite collection $\uparrow X_1, \dots, \uparrow X_n$, the union $X_1 \cup \cdots \cup X_n$ is contained in each upper set, i.e. contained in their intersection. Thus $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{U}$, and we form the ultraproduct $\mathbb{U} = \prod_{X \in I} \mathbb{A}_X / \mathcal{U}$. If γ is a sentence holding in $\mathbb{A}$, then it holds in $\mathbb{A}_{\{\gamma\}}$ and moreover in $\mathbb{A}_Y$ for any $Y \in \uparrow\{\gamma\}$. Then γ holds $\mathcal{U}$-almost everywhere, and γ is true in $\mathbb{U}$. Conversely, suppose γ does *not* hold in $\mathbb{A}$; then $\neg\gamma$ does, and $\neg\gamma \in T$. Thus $\neg\gamma$ holds in $\mathbb{A}_{\{\neg\gamma\}}$ and in every $\mathbb{A}_Y$ for $Y \in \uparrow\{\neg\gamma\}$, hence $\neg\gamma$ holds in $\mathbb{U}$, and γ does not hold in $\mathbb{U}$. Thus, $\mathbb{A}$ and $\mathbb{U}$ have the same theory, and are elementarily equivalent.

(3) $\Rightarrow$ (1). Suppose $\mathbb{A}$ is elementary equivalent to $\prod_{X \in I} \mathbb{A}_X / \mathcal{U}$ for $\mathcal{U}$ an ultrafilter on I, and each $\mathbb{A}_X$ finite. If γ is a sentence in the theory of finite L-structures, then γ holds in all $\mathbb{A}_X$, so it certainly holds in the ultraproduct and thus in $\mathbb{A}$ as well. $\square$