Model Theory
Assignment 2
Problem 14

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14. Let $\kappa < \lambda < \mu$ be infinite cardinals. Give an example of a structure of cardinality μ that has a substructure of cardinality κ but no substructure of cardinality λ.

Let M be a set of cardinality μ and let X be a subset of M of cardinality κ. Let $\mathcal{L} := \{f_{a,b} \mid a, b \in M - X\}$ be a language of unary function symbols. We will define a structure $\mathbb{M}$ in this language which has a substructure of cardinality κ but no substructure of cardinality λ. Let the underlying set of $\mathbb{M}$ be M and for every $a, b \in M - X$, define $f_{a,b}^{\mathbb{M}} : M \to M$ as follows. Given $x \in M$, let

$$f_{a,b}^{\mathbb{M}}(x) = \begin{cases} b & \text{if } x = a \\ a & \text{if } x = b \\ x & \text{else} \end{cases}$$

Overall, $\mathbb{M} = \langle M \ ; \ \{f_{a,b}^{\mathbb{M}} \mid a, b \in M - X\}\rangle$. Next, consider the substructure of $\mathbb{M}$ generated by X. Since each $f_{a,b}^{\mathbb{M}}$ fixes every element of X, this substructure has underlying set X. Thus, this substructure of $\mathbb{M}$ has size κ.

Now let $\mathbb{N}$ be a substructure of $\mathbb{M}$ with underlying set N of cardinality greater than or equal to λ. Since $\lambda > \kappa$ and $|X| = \kappa$, N must contain some point of M not belonging to X. Let $a \in N - X$ and let $b \in M - X$. Since $\mathbb{N}$ is a substructure of $\mathbb{M}$, it is closed under $f_{a,b}^{\mathbb{M}}$, so $f_{a,b}^{\mathbb{M}}(a) = b \in N$. Our choice of $b \in M$ was arbitrary, so we conclude $M - X \subseteq N$. Since μ is infinite and $\kappa < \mu$, we have that $|M - X| = |M|$. We therefore conclude that $|N| \geq |M - X| = \mu > \lambda$, and so $\mathbb{M}$ has no substructure of cardinality λ.