Model Theory Homework 2

Howie Jordan, Chase Meadors

Problem 5.

Claim The number of ultrafilters on an infinite set I is $2^{2^{|I|}}$.

Proof First, we show that the number of ultrafilters is no larger than $2^{2^{|I|}}$. To see this, note that an ultrafilter is a particular set of subsets of I satisfying certain properties, hence there can be no more ultrafilters than there are sets of subsets of I. Since the number of sets of subsets of I is given by $|\mathcal{P}(\mathcal{P}(I))| = 2^{2^{|I|}}$, this gives an upper bound on the possible number of ultrafilters.

Let $\mathcal{F} = \mathcal{P}_\omega(I)$ be the collection of finite subsets of I and let $\Phi = \mathcal{P}_\omega(\mathcal{P}_\omega(I))$ be the collection of finite subsets of $\mathcal{F}$. Note that since I is infinite, we have that $|I| = |\mathcal{F}| = |\Phi| = |\mathcal{F} \times \Phi|$. To see that there are at least $2^{2^{|I|}}$ ultrafilters, we show that we can associate a distinct ultrafilter on $\mathcal{F} \times \Phi$ to each set of subsets of I. Since $\mathcal{F} \times \Phi$ is in bijection with I, we have a bijection between ultrafilters on $\mathcal{F} \times \Phi$ and ultrafilters on I, so there are the same number of ultrafilters on either set.

For any arbitrary subset of I, say $J \subseteq I$, let

$$A_J = \{(f, \phi) \in \mathcal{F} \times \Phi \mid J \cap f \in \phi\}$$

that is, A_J is the set of all pairs (f, ϕ) where f is a finite subset of I and ϕ is a finite set of subsets of I containing $J \cap f$. Let $A_J^c = \mathcal{F} \times \Phi - A_J$ be the complement of A_J. Note that each A_J is not empty, since $J \cap \varnothing = \varnothing \in \{\varnothing\}$, the pair $(\varnothing, \{\varnothing\}) \in A_J$. We also have that each A_J^c is not empty, as for example $(\varnothing, \varnothing)$ is a pair of a finite subset of I and a finite set of finite subsets of I not containing the intersection of the first set with J.

Now let $S \subset \mathcal{P}(I)$ be an arbitrary set of subsets of I and let $\mathcal{A}_S = \{A_J \mid J \in S\} \cup \{A_J^c \mid J \notin S\}$. We claim that any such $\mathcal{A}_S$ has the Finite Intersection Property, so that they can be extended to ultrafilters.

Let $J_1, \ldots, J_N \in S$ and $K_1, \ldots, K_M \notin S$ be arbitrary, so that $A_{J_1}, \ldots, A_{J_N}, A_{K_1}^c, \ldots, A_{K_M}^c \in \mathcal{A}_S$ is an arbitrary finite collection of elements of $\mathcal{A}_S$. For $n \in \{1, \ldots, N\}$, let $(f_n, \phi_n) \in A_{J_n}$. Take $f = \cup_{n=1}^N (f_n)$ and note that f is finite. We wish to exclude the possibility that for some $n \in \{1, \ldots, N\}$ and $m \in \{1, \ldots, M\}$ we have that $J_n \cap f = K_m \cap f$. Whenever this does occur, since $J_n \neq K_m$ as $J_n \in S$ but $K_m \notin S$, we can take $x \in (J_n - K_m) \cup (K_m - J_n)$. Note that now $J_n \cap (f \cup \{x\}) \neq K_m \cap (f \cup \{x\})$ as one side contains x while the other does not. Let f' be the set f together with all such x as necessary. Let $\phi = \{J_1 \cap f', \ldots, J_N \cap f'\}$ and note that by construction no $K_m \cap f' \in \phi$. Hence, $(f', \phi) \in A_{J_n}$ for each n and $(f', \phi) \in A_{K_m}^c$ for each m, so that the intersection $A_{J_1} \cap \cdots \cap A_{J_N} \cap A_{K_1}^c \cap \cdots \cap A_{K_M}^c$ is nonempty.

Now let $R, S \subseteq \mathcal{P}(I)$ such that $R \neq S$, and let $\mathcal{U}_R$ and $\mathcal{U}_S$ be the respective extensions of $\mathcal{A}_R$ and $\mathcal{A}_S$ to ultrafilters. Note that since $R \neq S$, at least one of them is nonempty. It also possible that one could contain the other, but without loss of generality we may take R to be nonempty and such that there is some $J \in R$ such that $J \notin S$. Then $A_J \in \mathcal{U}_R$ as $A_J \in \mathcal{A}_R$ by definition of $\mathcal{A}_R$ and $\mathcal{U}_R$ extends $\mathcal{A}_R$. Note that since $\mathcal{U}_S$ is an ultrafilter, either $A_J \in \mathcal{U}_S$ or $A_J^c \in \mathcal{U}_S$ but not both. Since $J \notin S$, $A_J^c \in \mathcal{A}_S$ by definition of $\mathcal{A}_S$. As $\mathcal{U}_S$ extends $\mathcal{A}_S$, $A_J^c \in \mathcal{U}_S$ as well. Hence, $A_J \notin \mathcal{U}_S$, so that $\mathcal{U}_R \neq \mathcal{U}_S$. This shows an injection from the set of the subsets of $\mathcal{P}(I)$ to the set of ultrafilters on $\mathcal{F} \times \Phi$, so that the number of ultrafilters on $\mathcal{F} \times \Phi$ is at least $2^{2^{|I|}}$. By the bijection from $\mathcal{F} \times \Phi$ to I, we thus have at least $2^{2^{|I|}}$ many ultrafilters on I as well. Together with the first paragraph, we have that the number of ultrafilters on I must be exactly $2^{2^{|I|}}$. ■