13 Give an example of a non-definable subset of each of the following structures.

(a) An infinite "pure set". (This is a structure in the "empty signature", meaning the signature with no nonlogical symbols.)

Without Parameters: Let Y be a non-empty proper subset of X. Then Y is not definable in X without parameters.

Proof: Since the only relation on X is the equality relation, every bijection from X to itself is an isomorphism. Every definable subset is preserved setwise by an automorphism. Let $r \in Y \neq \emptyset$ and let $s \in X \setminus Y \neq \emptyset$. Then the function $f : X \to X$ defined by $f(r) = s, f(s) = r$, and $f(x) = x$ for $x \in X \setminus \{r, s\}$ is a bijection (and hence an automorphism) but does not fix Y. Hence Y is not definable.

With Parameters: Let X be an infinite pure set and let $Y \subset X$ be a subset that is infinite and co-infinite (i.e. $X \setminus Y$ is infinite). Then Y is not definable with parameters.

Proof: Assume for a contradiction that Y is defined by a formula $\varphi(x, y_1, \ldots, y_n)$. Then for some $B_1, \ldots, b_n \in X$,

$$Y = \{x \in X \,|\, \varphi(x, b_1, \ldots, b_n)\}.$$

Since X is a pure set, every bijection from X to itself is an automorphism. Let $r \in Y \setminus \{b_1, \ldots, b_n\} \neq \emptyset$ and let $s \in (X \setminus Y) \setminus \{b_1, \ldots, b_n\} \neq \emptyset$. Consider the function $f : X \to X$ defined by $f(r) = s, f(s) = r$, and $f(x) = x$ for $x \in X \setminus \{r, s\}$. Then f is an automorphism, and f fixes $\{b_1, \ldots, b_n\}$ pointwise. Hence by Proposition 1.3.5 in Marker, f fixes Y setwise. But $r \in Y$ and $s = f(r) \in X \setminus Y$, a contradiction. Therefore Y is not definable.

(b) The abelian group $\mathbb{Z} = \langle \text{integers}; +, -, 0 \rangle$.

Without Parameters: The singleton set $\{1\}$ is not definable in $\mathbb{Z}$ without parameters.

Proof: The map $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = -n$ is an automorphism of $\mathbb{Z}$. Since $\{1\}$ is not fixed by f, $\{1\}$ is not definable, as definable subsets are fixed by automorphisms.

With Parameters: The Diagonal Halting Problem, $\{n \in \mathbb{N} : \varphi_n(n) \text{ halts }\}$ is not definable in $\mathbb{Z}$ with parameters.

Proof: We expand our language to $\langle \text{integers}; +, -, 0, \cdot, 1 \rangle$. Then any set not definable in the expanded language will also not be definable in the original language. Gödel proved that sets definable in the ring of integers with parameters are exactly the sets that are recursive (i.e. there is a Turing Machine that decides membership in this set). It is well known that the Diagonal Halting Problem is not recursive, and hence it is not definable in the ring of integers. Hence the Diagonal Halting Problem is not definable in the original language, $\mathbb{Z}$, with parameters.

(c) The field $\mathbb{C} = \langle \text{complex numbers}; +, -, 0, \cdot, 1 \rangle$.

Without Parameters: The singleton set $\{i\}$ is not definable in $\mathbb{C}$ without parameters.

Proof: The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{z}$, the complex conjugate of z, is a field automorphism of $\mathbb{C}$. Since $f(i) = -i \neq i$, f does not fix $\{i\}$, but automorphisms fix definable subsets setwise. Hence $\{i\}$ is not definable in $\mathbb{C}$.

With Parameters: The set of transcendental numbers is not definable in $\mathbb{C}$ with parameters.

Proof: First notice that every term with parameters in $\mathbb{C}$ is a term in $\mathbb{C}_C$, and so it is a polynomial with complex coefficients. Since there are only operations and no relations in $\mathbb{C}$, every atomic formula with only one free variable is of the form $p(x) = q(x)$ for polynomial p and q with complex coefficients. Hence every atomic formula with one free variable is equivalent to an atomic formula of the form $p(x) = 0$ for some polynomial, p. Also, $p(x) = 0 \vee q(x) = 0$ is equivalent to $p(x) \cdot q(x) = 0$. So, since algebraically closed fields have quantifier elimination, every formula with one free variable is equivalent to a finite conjunction of terms of the form $p(x) = 0$ and $\neg p(x) = 0$.

Now, each non-zero polynomial in $\mathbb{C}$ has finitely many roots. So $\{x \in \mathbb{C} \,|\, p(x) = 0\}$ contains just finitely many transcendental numbers, and $\{x \in \mathbb{C} \,|\, \neg p(x) = 0\}$ contains all but finitely many transcendental numbers and all but finitely many algebraic numbers. There are infinitely many algebraic numbers and infinitely many transcendental numbers. Notice that

$$\{x | \neg p_1(x) = 0 \wedge \cdots \wedge \neg p_m(x) = 0\} = \{x | \neg(p_1(x) = 0 \vee \cdots \vee p_m(x) = 0\} = \{x | \neg(p_1 \cdot \ldots \cdot p_m)(x) = 0\}$$

contains infinitely many algebraic integers. Hence no formula with one free variable defines the set of transcendental numbers, and so the transcendental numbers are a non-definable subset of $\mathbb{C}$.