MODEL THEORY: ASSIGNMENT 2, PROBLEM 2

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Problem 2

Let L be a language and let X=Spec(L) be its space of complete theories. Show that for any ordinal α there is a theory T_α such that $V(T_\alpha)$ is the closed set of all complete L-theories of Cantor-Bendixson rank at least α. Describe how to generate a set of axioms for T_α.

If the set $V(T_\alpha) = \{T \mid T_\alpha \subseteq T\}$, where T is a complete theory, is to be the set of all complete theories with CB-rank at least α, then T_α must be a set of sentences such that all and only those complete theories containing T_α are of CB-rank at least α. For any theory T, T has CB-rank α if $T \in X^{(\alpha)}$ *and* $T \notin X^{(\alpha+1)}$. So, for T to have CB-rank of at least α all that need be is $T \in X^{(\alpha)}$. Since T_α must be defined for all ordinals, the description of how to axiomatize T_α will be broken up into the three cases: when $\alpha = 0$, when α is a successor ordinal, and when α is a limit ordinal. Take first the case where $\alpha = 0$.

The zero case: This case is somewhat trivial. Since every theory has CB-rank at least 0, what is wanted is a set of axioms that is included in every theory. Since $\emptyset \subseteq T$ for all theories T, $\emptyset$ axiomatizes all theories of CB-rank at least 0 (the whole of X). Now consider the case when α is a successor ordinal.

The successor ordinal case: If $T \in X^{(\alpha)}$ then, by the definition of the CB-derivative for successor ordinals, T is a limit point of the set $X^{(\alpha-1)}$. So, axioms for T_α must imply that any complete theory $T \supseteq T_\alpha$ is a limit point in $X^{\alpha-1}$. Since every point is a limit point or an isolated point but not both, the set of all limit points of $X^{(\alpha-1)}$ may be defined as the complement in $X^{(\alpha-1)}$ of the set of isolated points. A theory T is isolated in $X^{(\alpha-1)}$ if there is some open set O whose intersection with $X^{(\alpha-1)}$ is T. If O is an open set then O must be either in the basis of X or a union of basic sets. If O is the union of basic sets, then there is at least one set in said union that must have T as an element. Since the intersection of the union, O, with $X^{(\alpha-1)}$ is just T, the intersection of the basic set containing T with $X^{(\alpha-1)}$ will be just T as well. So, if T is an isolated point, then there is some set in the basis of X whose intersection with $X^{(\alpha-1)}$ is T. Since the basic sets of X are clopen sets of the form $O_\sigma = \{\ T \mid \sigma \in T\}$, the set of all isolated points of $X^{(\alpha-1)}$ would then be $(\bigcup_{\sigma\in\theta_{\alpha-1}} O_\sigma)$, where $\theta_{\alpha-1} = \{\sigma \mid O_\sigma \cap X^{(\alpha-1)} = \{T\}\}$ for some theory T. I.e. O_σ is a singleton in $X^{(\alpha-1)}$. The set of all limit points would then be $(\bigcup_{\sigma\in\theta_{\alpha-1}} O_\sigma)^C = \bigcap_{\sigma\in\theta_{\alpha-1}}(O_\sigma)^C = \bigcap_{\sigma\in\theta_{\alpha-1}}(O_{\neg\sigma})$. Since $O_{\neg\sigma} = \{\ T \mid \neg\sigma \in T\}$, any $T \in \bigcap_{\sigma\in\theta_{\alpha-1}}(O_{\neg\sigma})$ must have $\{\neg\sigma \mid \sigma \in \theta_{\alpha-1}\} \subseteq T$. Because the set of all limit points in $X^{(\alpha-1)}$ is $X^{(\alpha)}$, $T \in X^{(\alpha)}$ only when $\{\neg\sigma \mid \sigma \in \theta_{\alpha-1}\} \subseteq T$. However, since $\{\neg\sigma|\sigma \in \theta_{\alpha-1}\}$ only defines the limit points with respect to the set $X^{(\alpha-1)}$, it should be taken together with the axioms for $T_{\alpha-1}$ to create a set of sentences that define theories that are limit points for all X^β where $\beta \leq \alpha - 1$. So any $T \supseteq (\{\neg\sigma | \sigma \in \theta_{\alpha-1}\} \cup T_{(\alpha-1)})$ has CB-rank of at least α. Thus $\{\neg\sigma | \sigma \in \theta_{\alpha-1}\} \cup T_{(\alpha-1)}$ constitutes a set of axioms for T_α. Now let us consider when α is a limit ordinal.

The limit ordinal case: The CB-derivative is defined at any limit ordinal α as $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$. So, if $T \in X^{(\alpha)}$, then $T \in X^{(\beta)}$, for all $\beta < \alpha$. This means T is a limit point in every $X^{(\beta)}$, $\beta < \alpha$. But T is only a limit point in $X^{(\beta)}$ if $T_\beta \subseteq T$. So, $T \in X^{(\alpha)}$ if $T_\beta \subseteq T$ for all $\beta < \alpha$. This means that $\bigcup_{\beta<\alpha} T_\beta \subseteq T$ for any $T \in X^{(\alpha)}$. Thus, $\bigcup_{\beta<\alpha} T_\beta$ constitutes a set of axioms for T_α, for any limit ordinal α.

In conclusion, we have $\emptyset$ as the axioms for T_0 and $\{\neg\sigma | \sigma \in \theta_{\alpha-1}\} \cup T_{(\alpha-1)}$ as the axioms for T_α when α is a successor ordinal and $\bigcup_{\beta<\alpha} T_\beta$ as the axioms when α is a limit ordinal.