12. Let T be a theory in a countable language. Show that if T has an infinite model, then some countable model of T is not finitely generated.
[**Hint: Use elementary chains.**]

Since T has an infinite model, it also has an uncountable model $\mathcal{N}$. We will define an elementary chain $(\mathcal{M}_i, i \in \omega)$ of countably infinite submodels of $\mathcal{N}$ through recursion. First, let X be any countably infinite subset of $\mathcal{N}$. By the Löwenheim-Skolem theorem, there is an elementary submodel of $\mathcal{N}$ of countably infinite size containing X. Let $\mathcal{M}_0$ be such a submodel. Then let $i \in \omega$ and suppose $\mathcal{M}_i$ has already been defined. Since M_i is countable and N is uncountable, there is some element $x \in N - M_i$. Let $\mathcal{M}_{i+1}$ be any countable elementary submodel of $\mathcal{N}$ containing $M_i \cup \{x\}$, which again exists by the (downward) Löwenheim-Skolem theorem. By recursion, $(\mathcal{M}_i, i \in \omega)$ is a chain of countably infinite elementary submodels of $\mathcal{N}$. Moreover, the chain is elementary due to the following lemma:

Lemma: Let $\mathcal{L}$ be any language and let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be $\mathcal{L}$-structures with $A \subseteq B \subseteq C$. If $\mathcal{A} \preceq \mathcal{C}$ and $\mathcal{B} \preceq \mathcal{C}$, then $\mathcal{A} \preceq \mathcal{B}$.

Proof: Let $\phi(v, \overline{w})$ be a formula and let $\overline{a} \in A^n$. Let $b \in B$ and suppose $\mathcal{B} \models \phi(b, \overline{a})$. Since $\mathcal{B}$ is an elementary substructure of $\mathcal{C}$ (i.e., the inclusion map from B to A is an elementary embedding), we also have that $\mathcal{C} \models \phi(b, \overline{a})$. Moreover, $\mathcal{A}$ is an elementary substructure of $\mathcal{C}$, so the Tarski-Vaught test implies that there is some $a' \in A$ such that $\mathcal{A} \models \phi(a', \overline{a})$. Again by the Tarski-Vaught test, we now see that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. $\square$

To continue with the main proof, let $i, j \in \omega$ such that $i < j$. We have $M_i \subset M_j \subset N$, $\mathcal{M}_i \prec \mathcal{N}$, and $\mathcal{M}_j \prec \mathcal{N}$, which implies that $\mathcal{M}_i \prec \mathcal{M}_j$. Hence, $(\mathcal{M}_i, i \in \omega)$ is an elementary chain of countably infinite submodels of $\mathcal{N}$.

Now let $\mathcal{M} := \bigcup_{i \in \omega} \mathcal{M}_i$, which is countable since this is a countable union of countable sets. It is shown in the proof of Proposition 2.3.11 that

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{M}_i \models \phi(\overline{a})$$

for all $i \in \omega$, all formulas $\phi(\overline{v})$, and all $\overline{a} \in M_i^n$. This implies that $\mathcal{M}$ is a model of T since each $\mathcal{M}_i$ is a model of T. However, we claim that $\mathcal{M}$ is not finitely generated. Suppose for sake of contradiction that $\{a_1, ..., a_n\}$ is a generating set for $\mathcal{M}$. Let $k \in \omega$ such that $\{a_1, ..., a_n\} \subseteq M_k$ (such a value of k exists since each a_i must appear in *some* M_j). Thus, $\mathcal{M} = \langle \{a_1, ..., a_n\} \rangle \subseteq M_k$. However, this contradicts the fact that $(M_i, i \in \omega)$ is a strictly increasing chain! We therefore conclude that $\mathcal{M}$ is a countably infinite model of T which is not finitely generated. $\square$