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12. Explain why all of the following sets are definable in $\mathbb{R} = \langle\{\text{reals}\}; \cdot, +\rangle$.

(a) The unit interval

(b) $\{n\}$ for any integer n

(c) $\{x\}$ for any algebraic number x

Proof.

(a) We can define the singleton $\{1\}$ with the formula $\phi_1(x) = (\forall y)(x \cdot y = y)$. We can also define the singleton $\{0\}$ with the formula $\phi_0(x) = (\forall y)(x + y = y)$. Now define $\phi_{\leq}(x, y) = (\exists z)(y = x + (z \cdot z))$. Then the unit interval is defined by

$$\phi_{[0,1]}(x) = (\exists a_1)(\exists a_2)(\phi_0(a_1) \wedge \phi_1(a_2) \wedge \phi_{\leq}(a_1, x) \wedge \phi_{\leq}(x, a_2))$$

As a side note we show that we can also define strict inequality via the formula $\phi_<(x, y) = (\exists z)((y = x + (z \cdot z) \wedge \neg\phi_0(z))$. Then the open unit interval is defined via

$$\phi_{[0,1]}(x) = (\exists a_1)(\exists a_2)(\phi_0(a_1) \wedge \phi_1(a_2) \wedge \phi_<(a_1, x) \wedge \phi_<(x, a_2))$$

(b) We can define the singleton $\{n\}$ for a positive integer n with the formula $\phi_n(x) = (\exists a_1)(\phi_1(a_1) \wedge (x = a_1 + a_1 + \cdots + a_1))$, where a_1 is added n times. This is a finite formula and well-defined.

(c) We can now define an integer n. We have already defined the positive integers. If $n < 0$, we have already defined $-n$ so $\phi_n(x) = (\exists a_1)(\phi_{-n}(a_1) \wedge (x + a_1 = 0))$. We can also now define any rational number $\frac{m}{n}$, via $\phi_{\frac{m}{n}}(x) = (\exists a_1)(\exists a_2)(\phi_m(a_1) \wedge \phi_n(a_2) \wedge (x \cdot a_2 = a_1))$. Suppose x is an algebraic number. We now have the tools to define the set $\{x\}$. Since it is algebraic it satisfies some rational polynomial $r_n x^n + \cdots + r_1 x + r_0$. The roots of this polynomial, $\Lambda = \{x_1, \ldots, x_k\}$, is definable by the formula

$$\phi_\Lambda(x) = (\exists a_1), \ldots, (\exists a_n)(\phi_{r_0}(a_1) \wedge \cdots \wedge \phi_{r_n}(a_{n+1}) \wedge (a_1 + a_2 x + \cdot + a_{n+1} x \cdots x = 0))$$

If one is dealing with real roots, these roots are linearly ordered. Then one can specify the l-th root of the polynomial by the formula

$$\begin{aligned}\phi_{\Lambda_l}(x) = &(\exists b_1), \ldots, (\exists b_{k-1})\\ &(\phi_\Lambda(b_1) \wedge \cdots \wedge \phi_\Lambda(b_{k-1}) \wedge \phi_\Lambda(x)) \wedge\\ &(\phi_<(b_1, b_2) \wedge \phi_<(b_2, b_3) \wedge \cdots \wedge \phi_<(b_{l-1}, x) \wedge \phi_<(x, b_l) \wedge \cdots \wedge \phi_<(b_{k-2}, b_{k-1}))\end{aligned}$$

□