Model Theory Homework 3

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Problem 4.

Claim Let $\mathcal{A}$ be a finite L-structure. Then $Th(\mathcal{A})$ has quantifier elimination if and only if $\mathcal{A}$ is ultrahomogeneous[1]

Proof For the forward direction, suppose that $Th(\mathcal{A})$ has quantifier elimination. Let $\mathcal{B}$ and $\mathcal{C}$ be finitely generated substructures such that there is an isomorphism $f : \mathcal{B} \to \mathcal{C}$. We will extend f to an automorphism as follows.

Let $\bar{b} = (b_1, \ldots, b_k)$ be an enumeration of $\mathcal{B}$. Then $\bar{c} = (f(b_1), \ldots, f(b_k))$ gives an enumeration of $\mathcal{C}$. Since f is an isomorphism, $\mathrm{tp}^{\text{q.f.}}(\bar{b}) = \mathrm{tp}^{\text{q.f.}}(\bar{c})$. Since $Th(\mathcal{A})$ has quantifier elimination, this implies that $\mathrm{tp}(\bar{b}) = \mathrm{tp}(\bar{c})$.

Consider the structures $\mathcal{A}_{\bar{b}}$ and $\mathcal{A}_{\bar{c}}$, where we expand the language L by k constants, say $a_1, \ldots, a_k$ and interpret them in the first case as $b_1, \ldots, b_k$ and in the second as $c_1, \ldots, c_k$. Since $\mathrm{tp}(\bar{b}) = \mathrm{tp}(\bar{c})$, we have that the structures $\mathcal{A}_{\bar{b}}$ and $\mathcal{A}_{\bar{c}}$ are elementarily equivalent finite structures in the expanded language. Since all sentences true in one are true in the other, by the contrapositive of Assignment 1 Problem 11[2] we have that the $\mathcal{A}_{\bar{b}}$ is isomorphic to $\mathcal{A}_{\bar{c}}$. Let F be the isomorphism exhibiting this. Then $F(\bar{b}) = \bar{c}$ as it must preserve the interpretation of the constant symbols, so this isomorphism extends f. But then, regarding F as a map on the structure $\mathcal{A}$ we have an automorphism of $\mathcal{A}$ that extends f, as desired.

Now for the reverse direction, suppose that $\mathcal{A}$ is ultrahomogeneous. Suppose towards a contradiction that $Th(\mathcal{A})$ does not have quantifier elimination. Quantifier elimination of a theory is equivalent to having quantifier elimination locally at each type, i.e. that any tuples with equal quantifier free types must have equal full types[3]. Hence, if $Th(\mathcal{A})$ does not have quantifier elimination then there must exist n-tuples $\bar{b}$ and $\bar{c}$ of elements of A such that $\mathrm{tp}^{\text{q.f.}}(\bar{b}) = \mathrm{tp}^{\text{q.f.}}(\bar{c})$ but $\mathrm{tp}(\bar{b}) \neq \mathrm{tp}(\bar{c})$.

Consider the substructures $\mathcal{B}$ and $\mathcal{C}$, generated respectively by $\bar{b}$ and $\bar{c}$. Since $\mathrm{tp}^{\text{q.f.}}(\bar{b}) = \mathrm{tp}^{\text{q.f.}}(\bar{c})$, the generated structures have the same size and the map f extending $\bar{b} \mapsto \bar{c}$ is an isomorphism. By ultrahomogeneity, f extends to an automorphism of all of $\mathcal{A}$ which maps $\bar{b}$ to $\bar{c}$. However, automorphisms preserve type, so in fact $\mathrm{tp}(\bar{b}) = \mathrm{tp}(\bar{c})$, a contradiction. Hence, $Th(\mathcal{A})$ must have quantifier elimination. ■

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[1] We say a structure is ultrahomogenous when every isomorphism between any finitely generated substructures lifts to automorphism of the whole structure.
[2] Thanks to Meredith, Murali, and Muro.
[3] As a corollary of Theorem 3.1.4, page 73 in Marker