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7. Let L be the language whose only nonlogical symbol is one unary relation symbol. Find all L-theories that have quantifier elimination. Are they all complete?

Proof. Let R denote the unary relation symbol in the language L. To any countable L-structure A we can associate a pair of cardinals $(m, n) = (|R[A]|, |A - R[A]|)$, which we will call the *characteristic* of A, where m and n are either finite or ω. We will prove the following lemma.

Lemma 1. *An L-theory T has quantifier elimination if and only if whenever $B, C \models T$ are models with isomorphic 1-element substructures, then B and C must have the same characteristic.*

We will use the fact (Theorem 3.1.4 in *Marker*) that T has quantifier elimination if and only if

$$\mathrm{tp}_B^{q.f.}(\mathbf{b}) = \mathrm{tp}_C^{q.f.}(\mathbf{c}) \longrightarrow \mathrm{tp}_B(\mathbf{b}) = \mathrm{tp}_C(\mathbf{c})$$

for $\mathbf{b} \in B^n, \mathbf{c} \in C^n$. To show the forward direction, let $B' \subseteq B, C' \subseteq C$ be isomorphic substructures generated by the elements $b \in B$ and $c \in C$ respectively (if no such substructures exist we are done), then we have $\mathrm{tp}_B(b) = \mathrm{tp}_C(c)$. Let $\sigma^{\pm}_{(n,m)}(x)$ be the sentence

$$\sigma^{\pm}_{(n,m)}(x) = \pm R(x) \wedge \left((\exists y_1) \cdots (\exists y_{n+m}) \left(\bigwedge_{i,j} (y_i \neq y_j) \wedge \bigwedge_{i=1}^{n} R(y_i) \wedge \bigwedge_{i=n+1}^{n+m} \neg R(y_i) \right) \right)$$

Then we can see that $\sigma^{\pm}_{(n,m)}(x) \in \mathrm{tp}_B(b)$ if and only if $\pm R(b)$ and $n \leq |R[B]|$, $m \leq |B - R[B]|$. If $\sigma^{\pm}_{(n,m)}(x)$ is in $\mathrm{tp}_B(b)$ for all n that would imply that $|R[B]| = \omega$, and similar reasoning would apply if $|B - R[B]| = \omega$. Since $\mathrm{tp}_B(b) = \mathrm{tp}_C(c)$ this means B and C must have the same characteristic.

For the other direction, we may assume that B and C have isomorphic 1-element substructures and thus have the same characteristic (if they don't have isomorphic substructures there is nothing to check). To show that that T has quantifier elimination, we must show that if $\mathbf{b}$ and $\mathbf{c}$ have the same quantifier free type then they have the same elementary type, for $\mathbf{b} \in B^n, \mathbf{c} \in C^n$. Let B' be the substructure generated by $\mathbf{b}$ and C' the substructure generated by $\mathbf{c}$. Note that in our case a substructure is a subset with R restricted to that subset, so B' and C' have as their underlying sets the elements in the tuples $\mathbf{b}$ and $\mathbf{c}$ respectively. Also note that $\mathrm{tp}_B^{q.f.}(\mathbf{b})$ contains the sentence

$$\bigwedge_{i \in U} R(x_i) \wedge \bigwedge_{j \notin U} \neg R(x_j)$$

for a unique $U \subseteq \{1, \ldots, n\}$ determined by the tuple $\mathbf{b}$, and so $\mathrm{tp}_B^{q.f.}(\mathbf{b})$ determines B' up to isomorphism.

Since $\mathrm{tp}_B^{q.f.}(\mathbf{b}) = \mathrm{tp}_C^{q.f.}(\mathbf{c})$, we must have that $B' \cong C'$, and consequently the characteristic of B' must be the same as the characteristic of C'. Let $f : B' \to C'$ be an isomorphism of B' and C', then we can extend f to an isomorphism $\tilde{f} : B \to C$ since $|(B')^c \cap R[B]| =$

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$|(f(B'))^c \cap R[C]|$ and $|(B')^c \cap (B - R[B])| = |(f(B'))^c \cap (C - R[C])|$. Hence $\mathrm{tp}_B(\mathbf{b}) = \mathrm{tp}_C(\mathbf{c})$, and so we can conclude that T has quantifier elimination.

$\square$

We may now ask which of the L-theories with quantifier elimination, as described above, are complete?

Lemma 2. *A non-complete L-theory with quantifier elimination must have exactly two models up to equivalence which have characteristics $(n, 0)$ and $(0, m)$ for some n and m.*

Proof. If T is an L-theory which is not complete and has quantifier elimination, then T has at least two models that are not elementary equivalent, meaning they cannot have the same characteristic. Since models which have isomorphic 1-element substructures must have the same characteristic per the previous lemma, these two models, which we will call A and B, must satisfy $\forall x\, R(x)$ and $\forall x\, \neg R(x)$, respectively. Then the characteristic of A is $(n, 0)$ and the characteristic of B is $(0, m)$ for some n and m. Any other model M must have the same characteristic as A if it satisfies $\exists x\, R(x)$, since M and A would have an isomorphic 1-element substructure. Similarly any model satisfying $\exists x\, \neg R(x)$ must have the same characteristic as B, hence there are exactly two models of T up to elementary equivalence.

$\square$