MODEL THEORY
HOMEWORK ASSIGNMENT II

Read Chapter 2.

PROBLEMS

1. (Toby Aldape, Sangman Lee, Vishnu Murali) Determine all languages in countable signatures which have the property that their space of complete theories is scattered.

(Shortcuts and hints: observe that any function or relation of positive arity can 'simulate' a constant, that any function or relation can simulate one of lower arity, that $(n+1)$-ary relations can simulate n-ary functions, while n-ary functions can usually simulate n-ary relations. Then consider these special cases of the problem: Case 1. a signature of infinitely many constant symbols. Case 2. A signature of one unary function symbol.)

2. (Raymond Baker, Thomas Magnuson, Mateo Muro) Let L be a language and let $X = \mathrm{Spec}(L)$ be its space of complete theories. Show that for any ordinal α there is a theory T_α such that $V(T_\alpha)$ is the closed set of all complete L-theories of Cantor-Bendixson rank at least α. Do this by describing how to generate a set of axioms for T_α.

3. (Chris Eblen, Trevor Manders, Andrew Stocker) Let $n > 0$ be chosen and fixed. Let L be the language of one binary predicate, $E(x, y)$, and let T be the L-theory of one equivalence relation with n classes. That is, T is axiomatized by the sentences asserting that E defines a reflexive, symmetric, transitive binary relation, along with a sentence

$$(\exists x_1)\cdots(\exists x_n)\left(\left(\bigwedge_{i\neq j}\neg E(x_i,x_j)\right)\wedge(\forall y)\left(\bigvee_{i}E(x_i,y)\right)\right)$$

which asserts that E has exactly n classes.

The purpose of this problem is to investigate the Cantor-Bendixson rank of the closed subset $V(T) \subseteq \mathrm{Spec}(L)$ consisting of the complete theories that extend T.

(a) Describe the complete theories of Cantor-Bendixson rank 0 in $V(T)$.
(b) Describe the complete theories of Cantor-Bendixson rank 1 in $V(T)$.
(c) Make a conjecture about the Cantor-Bendixson rank of $V(T)$.

4. (Hayden Hollis, Connor Meredith, Patrick Wynne) Let Th_L be the lattice of all L-theories for some language L. Show that any atom in this lattice has a complement, any complement of an atom is a coatom (and vice versa), but that there must exist at least one coatom that does not have a complement.

5. (Howie Jordan, Chase Meadors) Show that the number of ultrafilters on an infinite set I is $2^{2^{|I|}}$, by proving the following two statements.

(a) # ultrafilters $\leq 2^{2^{|I|}}$.
(b) # ultrafilters $\geq 2^{2^{|I|}}$.

Hints for (b): Let $\mathcal{F}$ be the set of finite subsets of I and let Φ be the set of finite subsets of $\mathcal{F}$. For each subset $J \subseteq I$ define

$$A_J = \{(f, \phi) \in \mathcal{F} \times \Phi \mid J \cap f \in \phi\}.$$

Let $A_j^c := (\mathcal{F} \times \Phi) - A_j$ be the complement of A_J. For each subset $S \subseteq \mathcal{P}(I)$, show that the set $\mathcal{A}_S = \{A_J \mid J \in S\} \cup \{A_J^c \mid J \notin S\}$ has the Finite Intersection Property. Thus each $\mathcal{A}_S$ extends to some ultrafilter $\mathcal{U}_S$. On the other hand, prove that $R \neq S$ implies $\mathcal{U}_R \neq \mathcal{U}_S$ by arguing that if $R, S \subseteq \mathcal{P}(I)$ and $J \in R - S$, then $A_J \in \mathcal{U}_R - \mathcal{U}_S$.

6. (Toby Aldape, Sangman Lee, Vishnu Murali) An (nonprincipal) ultrafilter is *uniform* if all of its sets have the same size. Show that a regular ultrafilter is uniform.

7. (Raymond Baker, Thomas Magnuson, Mateo Muro) Let L be a language in a signature $(\mathcal{F}, \mathcal{R}, \mathrm{ar})$. A class of L-structures is pseudo-elementary if it is elementary in some richer signature. ("Richer" means "richer or equal".)

(a) Show that, in the language of one unary predicate $P(x)$, the class of structures $\langle A; P(x)\rangle$ where $|P[A]| = |A - P[A]|$ is pseudo-elementary but not elementary.
(b) Show that pseudo-elementary classes are closed under ultraproducts.

8. (Chris Eblen, Trevor Manders, Andrew Stocker) Show that an ultraproduct of finite structures over the index set $I = \omega$ is either finite or of size $2^{\aleph_0}$.

9. **(Hayden Hollis, Connor Meredith, Patrick Wynne)**

(a) Let $\mathcal{U}$ be an ultrafilter on a set I, and let $\{\mathbb{A}_i \mid i \in I\}$ be a set of L-structures. Show that $\prod_{\mathcal{U}} \mathbf{Aut}(\mathbb{A}_i)$ is embeddable in $\mathbf{Aut}(\prod_{\mathcal{U}} \mathbb{A}_i)$. (Elements of $\prod_{\mathcal{U}} \mathbf{Aut}(\mathbb{A}_i)$ are called internal automorphisms of $\prod_{\mathcal{U}} \mathbb{A}_i$, while other automorphisms are called external.)

(b) Give an explicit example of an external automorphism.

10. **(Howie Jordan, Chase Meadors) Show that the following conditions on $\mathbb{A}$ are equivalent.**

(a) $\mathbb{A}$ is a model of the theory of the class of finite L-structures.

(b) Every L-sentence true in $\mathbb{A}$ holds in some finite L-structure.

(c) $\mathbb{A}$ is elementarily equivalent to an ultraproduct of finite L-structures.

($\mathbb{A}$ is *pseudofinite* if these hold.)

11. **(Toby Aldape, Sangman Lee, Vishnu Murali) Look up the definitions of Σ^1_1 and Π^1_1 sentences if necessary, and include the definitions in your solution to the following problems.**

(a) Show that ultraproducts preserve Σ^1_1-sentences.

(b) Give an example of a Π^1_1-sentence not preserved by ultraproducts.

12. **(Raymond Baker, Thomas Magnuson, Mateo Muro) Let $\mathcal{K}$ be a class of L-structures. Show that an ultraproduct of ultraproducts of members of $\mathcal{K}$ is isomorphic to an ultraproduct of members of $\mathcal{K}$.**

13. **(Chris Eblen, Trevor Manders, Andrew Stocker) Show that any structure is embeddable in an ultraproduct of its finitely generated substructures. Conclude that any universal class is generated by its finitely generated members. Show that this statement about universal classes is not true for arbitrary elementary classes. (Universal class = a class axiomatizable by universally quantified sentences.)**

14. **(Hayden Hollis, Connor Meredith, Patrick Wynne) Let $\kappa < \lambda < \mu$ be infinite cardinals. Give an example of a structure of cardinality μ that has a substructure of cardinality κ but no substructure of cardinality λ.**