Model Theory
Assignment 3
Problem 1

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1. For each triple $(x, y, z) \in \{\text{elementary}, \text{not}\}^3$ find an example

$$\mathbb{A} \xrightarrow{f} \mathbb{B} \xrightarrow{g} \mathbb{C}$$

realizing the triple, if possible, or explain why there is no example.

Proof. We begin with an example of (elementary, elementary, elementary). Let $\mathbb{A} = \mathbb{Q}$, $\mathbb{B} = \mathbb{Q}[\sqrt{2}]$ and $\mathbb{C} = \mathbb{R}$, all in the language of their usual orderings. Let $f, g, g \circ f$ all be the inclusion maps. All three structures are dense linear orderings without endpoints, so it follows that the maps are elementary embeddings.

We cannot make find an example of (elementary, elementary, not). For suppose we had such a triple. That means for every first order σ-formula $\phi(x_1, \ldots, x_n)$ and all $(a_1, \ldots, a_n) \in A$, we have $\mathbb{A} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathbb{B} \models \phi(f(a_1), \ldots, f(a_n))$ if and only if $\mathbb{C} \models \phi(g(f(a_1)), \ldots, g(f(a_n)))$. Hence $g \circ f$ is forced to be an elementary embedding.

An example of (elementary, not, elementary) is provided. Let $\mathbb{A} = \mathbb{N}$, in the language of ordered sets. Now since $\mathbb{N}$ has infinite cardinality, there exist proper elementary extensions. We let $\mathbb{B}$ be such an extension, extended by the map f. Every element of A is definable. For example, $\phi_0(x) = (\forall y)(x \neq y \implies (x < y)$. Similarly $\phi_1(x) = (\forall y)(x \neq y \implies (y < x \implies \phi_0(y)))$ and you can continue on this way for all the natural numbers. This means $\mathbb{B}$ contains a copy of $\mathbb{N}$. But $\mathbb{B}$ is a proper extension so there must exist some $b \in B \setminus A$. Since b cannot be the n-th smallest element, it is "infinitely large", i.e. $b > f(k)$ for any $k \in \mathbb{N}$. However, $\mathbb{A} \models (\forall x)[x \neq 0 \implies (\exists y)(\exists z)(y < x \wedge x < z \wedge (\forall u)(u < x \implies (u < y \vee u = y)) \wedge (\forall v)(x < v \implies (v > x \vee v = z)))]$. That means for all $b \in \mathbb{B} \setminus \mathbb{A}$, b has an immediate predecessor and immediate successor, so each b lives in some copy of $\mathbb{Z}$. Let $\mathbb{C} = \mathbb{B}$ be the same structure.

Now fix $b_0 \in \mathbb{B} \setminus \mathbb{A}$. As mentioned, the successor of b_0 and successor of successor of b_0 and so on combined with the predecessors make a copy of $\mathbb{Z}$, which will call $\mathbb{Z}_0$. We will let g be a map that is the identity everywhere except at $\mathbb{Z}_0$, where $g(b_0 + k) = b_0 + 2k$. Notice that $g \circ f$ is an elementary embedding, since $g \circ f = f$ and f was chosen to be elementary. $\mathbb{B} = \mathbb{C}$ to be an elementary extension, i.e. satisfying the same first order formulas. However, g itself is not an elementary embedding, for $\mathbb{B} \models (\neg\exists x)(b_0 < x \wedge x < b_0 + 1)$ but $\mathbb{C} \not\models (\neg\exists x)(b_0 < x \wedge x < b_0 + 2)$ since $b_0 + 1$ would be such an x.

An example of (elementary, not, not) is provided. Let $\mathbb{A} = \mathbb{Q}$, $\mathbb{B} = \mathbb{R}$, and $\mathbb{C} = \mathbb{R} \cup \infty$ in the language of orderings. Let $f, g, g \circ f$ all be inclusion mappings. Then f is an elementary embedding. g is not an elementary embedding since $\mathbb{B} \models (\neg\exists x)(\forall y)(x \geq y)$ but $\mathbb{C} \not\models (\neg\exists x)(\forall y)(x \geq y)$.

An example of (not, elementary, elementary) is not possible. Suppose such a triple existed. Then for every first order σ-formula $\phi(x_1, \ldots, x_n)$ and all $(a_1, \ldots, a_n) \in A$, we have $\mathbb{A} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathbb{C} \models \phi(g(f(a_1)), \ldots, g(f(a_n)))$ if and only if $\mathbb{B} \models \phi(f(a_1), \ldots, f(a_n))$. The two "if and only if"'s are because $g \circ f$ and g are elementary embeddings, respectively. All together this says that f is an elementary embedding.

An example of (not, elementary, not) is provided. Let $\mathbb{A} = \mathbb{N}$, $\mathbb{B} = \mathbb{Q}$, and $\mathbb{C} = \mathbb{R}$, all in the language of orderings. Let all three functions be inclusion maps. We know g is an elementary embedding. Let $\phi(x)$ be the formula $\forall y(y \neq x \implies y > x)$ and let 0 be the usual number zero. Then we know that $\mathbb{A} \models \phi(0)$ but $\mathbb{B} \not\models \phi(f(0))$ and $\mathbb{C} \not\models \phi(g(f(0))$.

An example of (not, not, elementary) is provided. Let $\mathbb{A} = \mathbb{C} = \mathbb{R}$ and let $\mathbb{B} = [-\pi/2, \pi/2]$ in their usual orderings. Let f be the arctan function, g be the inclusion map. Then $g \circ f$ is just the arctan function into $\mathbb{R}$. From elementary calculus we know arctan to be an increasing function so it is injective and order preserving, making $g \circ f$ an embedding. Since the theory of dense linear orderings without endpoints has quantifier elimination we have an elementary embedding. However f and g are not elementary embeddings. f fails to preserve unboundedness and likewise g fails to preserve boundedness. That is $\mathbb{A} \models (\neg\exists x)(\forall y)(x \geq y)$, $\mathbb{B} \not\models (\neg\exists x)(\forall y)(x \geq y)$, and $\mathbb{C} \models (\neg\exists x)(\forall y)(x \geq y)$

Finally, an example of (not, not, not). Let $\mathbb{A} = \mathbb{Z}$, $\mathbb{B} = \mathbb{R}$ and $\mathbb{C} = \mathbb{R} \cup \infty$ in the usual language of ordering. Let the three maps be inclusion. Let $\phi_1(x, y) = \exists z(x < z \wedge z < y)$. Then $\mathbb{A} \not\models \phi_1(0, 1)$ but $\mathbb{B} \models \phi_1(f(0), f(1))$, so f is not an elementary embedding. Let $\phi_2 = \exists x \forall y(y < x \vee y = x)$. Then $\mathbb{A} \not\models \phi_2$, $\mathbb{B} \not\models \phi_2$, and $\mathbb{C} \models \phi_2$, so g and $g \circ f$ are not elementary embeddings.

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