Connor Meredith
Vishnu Murali
Mateo Muro

Exercise 1.4.2(b): Let $\mathcal{L}$ be any finite language and let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that there is an $\mathcal{L}$-sentence ϕ such that $\mathcal{N} \models \phi$ if and only if $\mathcal{N} \cong \mathcal{M}$.

We build up ϕ in several steps. Suppose $\mathcal{M}$ has size n and write $M = \{a_1, a_2, ..., a_n\}$. For each k-ary operation symbol o in $\mathcal{L}$, k-ary relation symbol r in $\mathcal{L}$, and variable symbols $x_{i_1}, ..., x_{i_k}, x_{i_{k+1}}$, define formulae

$$\phi_o(x_{i_1}, ..., x_{i_k}, x_{i_{k+1}}) := \begin{cases} o(x_{i_1}, ..., x_{i_k}) = x_{i_{k+1}} & \text{if } o^{\mathcal{M}}(a_{i_1}, ..., a_{i_k}) = a_{i_{k+1}} \\ o(x_{i_1}, ..., x_{i_k}) \neq x_{i_{k+1}} & \text{else} \end{cases}$$

and

$$\phi_r(x_{i_1}, ..., x_{i_k}) := \begin{cases} r(x_{i_1}, ..., x_{i_k}) & \text{if } r^{\mathcal{M}}(a_{i_1}, ..., a_{i_k}) \\ \neg r(x_{i_1}, ..., x_{i_k}) & \text{else} \end{cases}$$

Observe that each of the given formulae records partial information about the structure of $\mathcal{M}$. Next, define

$$\phi_o := \bigwedge_{x \in \{x_1, ..., x_n\}^{k+1}} \phi_o(x)$$

and

$$\phi_r := \bigwedge_{x \in \{x_1, ..., x_n\}^{k}} \phi_r(x)$$

Observe that since M is finite, these formulae are all finite. For each constant symbol c, let $\phi_c := (c = x_i)$ where $c^{\mathcal{M}} = a_i$. Finally, define

$$\phi := (\exists x_1) \cdots (\exists x_n) \left[\left(\bigwedge_{s \in \mathcal{L}} \phi_s \right) \wedge \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \right) \wedge \left((\forall x_{n+1}) \bigvee_{1 \leq i \leq n} x_{n+1} = x_i \right) \right]$$

Since $\mathcal{L}$ is finite, so too is ϕ. We have constructed ϕ to explicitly record all structural information about $\mathcal{M}$, and $\mathcal{M} \models \phi$ since $(a_1, ..., a_n)$ is an instance of ϕ.

Let $\mathcal{N}$ be an $\mathcal{L}$ structure. Isomorphic structures satisfy the same sentences, so we need only show that if $\mathcal{N} \models \phi$, then $\mathcal{N} \cong \mathcal{M}$. Suppose $\mathcal{N} \models \phi$ and let $(b_1, ..., b_n)$ be an instance of ϕ in $\mathcal{N}$. Then $N = \{b_1, ..., b_n\}$. Define $f : \mathcal{M} \to \mathcal{N} : a_i \mapsto b_i$. We will show that f is an isomorphism.

- Suppose o is a k-ary operation symbol in $\mathcal{L}$ and let $(a_{i_1}, ..., a_{i_k}, a_{i_{k+1}}) \in M^{k+1}$. Suppose $o^{\mathcal{M}}(a_{i_1}, ..., a_{i_k}) = a_{i_{k+1}}$. Then $\phi_o(x_{i_1}, ..., x_{i_k}, x_{i_{k+1}}) = [o(x_{i_1}, ..., x_{i_k}) = x_{i_{k+1}}]$. Since $(b_1, ..., b_n)$ is an instance of ϕ in $\mathcal{N}$, we now have that $o^{\mathcal{N}}(b_{i_1}, ..., b_{i_k}) = b_{i_{k+1}}$. By definition of f,

$$o^{\mathcal{N}}(f(a_{i_1}), ..., f(a_{i_k})) = o^{\mathcal{N}}(b_{i_1}, ..., b_{i_k}) = b_{i_{k+1}} = f(a_{i_{k+1}}) = f(o^{\mathcal{M}}(a_{i_1}, ..., a_{i_k}))$$

- Suppose r is a k-ary relation symbol in $\mathcal{L}$ and let $(a_{i_1}, ..., a_{i_k}) \in M^k$. Since $(b_1, ..., b_n)$ is an instance of ϕ in $\mathcal{N}$, and by definition of ϕ_r, we have that $r^{\mathcal{M}}(a_{i_1}, ..., a_{i_k})$ holds if and only if $r^{\mathcal{N}}(b_{i_1}, ..., b_{i_k})$ holds. Equivalently, $r^{\mathcal{M}}(a_{i_1}, ..., a_{i_k})$ holds if and only if $r^{\mathcal{N}}(f(a_{i_1}), ..., f(a_{i_k}))$ holds.

- Suppose c is a constant symbol in $\mathcal{L}$ and suppose $c^{\mathcal{M}} = a_i$. Since $(b_1, ..., b_n)$ is an instance of ϕ in $\mathcal{N}$ and $\phi_c = (c = x_i)$, b_i is an instance of ϕ_c in $\mathcal{N}$. Hence, $c^{\mathcal{N}} = b_i$, and so $f(c^{\mathcal{M}}) = f(a_i) = b_i = c^{\mathcal{N}}$.

Hence, f is a strong homomorphism. Since it is also bijective, it is an isomorphism.