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13. Let T be the theory of $A = \langle \omega; \cdot, + \rangle$ and show that $|S_1(T)| = 2^\omega$. Conclude that there are 2^ω countable models of T up to isomorphism.

Proof. We first show that $S_1(T)$ has size continuum and then that T has continuumly many models.

To start, we define formulae $\phi_0(x) = (\forall y)(x + y = y)$ and $\phi_1(x) = (\forall y)(x \cdot y = y)$. Then for every natural number n, greater than 1 we define the formula $\phi_n(x) = (\exists y)(\phi_1(y) \wedge (x = \underbrace{y + y + \ldots + y}_{n \text{ times}}))$. This is true if and only if x is n. Furthermore this is well formed as it is finite for any n. We define the formula $\phi_{div}(x, y) = (\exists z)(x \cdot z = y)$ which is true if x divides y. Lastly we use the abbreviation $\phi_{div}(n, x)$ to mean the formula $(\exists y)(\phi_{div}(y, x) \wedge \phi_n(y))$.

Let $s \in \{0, 1\}^\omega$ be a $0, 1$ sequence of length ω. We associate s with a partial 1-type $p_s(x)$ of the following form. Let q_i be the ith prime. If the ith symbol in the sequence is 1 we consider the formula $\phi_{div}(q_i, v)$ to be in $p_s(x)$ and if this is 0 we consider $\neg \phi_{div}(q_i, v)$ to be in $p_s(x)$. For example, if $s = (1, 1, 0, 0, \ldots)$ then we consider $p_s(v) = \{\phi_{div}(2, v), \phi_{div}(3, v), \neg \phi_{div}(5, v), \neg \phi_{div}(7, v), \ldots\}$. Given a finite subset F of $p_s(v)$ we can find a natural number that satisfies it by multiplying all the primes q_i, such that $\phi_{div}(q_i, v)$ appears in F, if no such formula occurs in F, then we consider the natural number 1 as this is not divisible by any of the primes. Thus we can always find a natural number satisfying all the formulae in F. Therefore $p_s(v)$ must be consistent.

Now we consider two partial types $p_s(v)$ and $p_t(v)$. If s and t are different then there must be some coordinate i where the two types differ. So it must be the case that the ith formula of $p_s(v)$ is the negation of the ith formula of $p_t(v)$. Any complete type q which contains both $p_s(v)$ and $p_t(v)$ must have both a formula and its negation, but this is impossible. Therefore we can conclude that no two partial types can extend to the same complete type, thus $S_1(T)$ is at least as large as the set of partial types. Thus $|S_1(T)| \geq 2^\omega$ as we have 2^ω many $\{0, 1\}$ sequences of length ω.

We know that the size of the scattered part of $S_1(T)$ must be less than or equal to ω and we have at least 2^ω many points in $S_1(T)$, therefore the size of the Perfect kernel must be 2^ω by the Cantor Bendixson Theorem. Therefore we have $|S_1(T)| = 2^\omega$.

Let us assume that the number of isomorphism types of countable models of T be κ. We must have $2^\omega \leq \kappa \omega$. Therefore $2^\omega \leq \kappa$. We must also have $\kappa \leq 2^\omega$ as there are 2^ω many ways of defining the addition and multiplication tables. Thus $\kappa = 2^\omega$. We conclude that there are 2^ω countable models of T up to isomorphism. □