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5. An L-theory T is substructure complete if whenever $\mathbb{A} \leq \mathbb{B}$ and $\mathbb{B} \vDash T$, then $T \cup \mathrm{Diag}(\mathbb{A})$ axiomatizes a complete $L(A)$-theory. Prove that T has quantifier elimination iff it is substructure complete.

Proof. ($\Longrightarrow$) Assume T has quantifier elimination. Let $\mathbb{A}$ be a substructure of a model $\mathbb{B}$ of T and take any $L(A)$-formula φ.

Our goal is to show that any $L(A)$-sentence $\varphi(\overline{a})$, which is obtained from evaluating the formula φ at $\bar{a} \in \mathbb{A}$, is decided by $T \cup \mathrm{Diag}(\mathbb{A})$. Suppose not. Then T and $\mathrm{Diag}(\mathbb{A})$ both do not decide $\varphi(\overline{a})$. In other words, T and $\mathbb{A}$ both do not decide $\varphi(\overline{a})$. So, there exist two super models $\mathbb{A} \hookrightarrow \mathcal{M}, \mathcal{N} \vDash T$ such that $\mathcal{M} \vDash \varphi(\overline{a})$ and $\mathcal{N} \vDash \neg\varphi(\overline{a})$. Since T has quantifier elimination, there is some quantifier-free $L(A)$-formula ψ such that $\mathcal{M} \vDash \varphi(\overline{a}) \leftrightarrow \psi(\overline{a})$ and $\mathcal{N} \vDash \neg\varphi(\overline{a}) \leftrightarrow \neg\psi(\overline{a})$. Any embedding between L-structures preserves and reflects quantifier-free formulae (cf. Prop 1.1.8 in Marker's). Hence, $\mathbb{A} \vDash \psi(\overline{a}), \neg\psi(\overline{a})$. This is a contradiction to the fact $\mathbb{A}$ is a model. Therefore, $T \cup \mathrm{Diag}(\mathbb{A})$ axiomatizes a complete $L(A)$-theory. □

Proof. ($\Longleftarrow$) Assume T is substructure complete. Let φ be a L-formula. Our goal is to show $T \vDash \forall\bar{v}\,(\varphi(\bar{v}) \leftrightarrow \psi(\bar{v}))$ for some quantifier-free L-formula ψ. So consider the type $\Gamma(\bar{v}) = \{\psi(\bar{v}) : \psi$ is quantifier-free and $T \vDash \forall\bar{v}(\varphi(\bar{v}) \rightarrow \psi(\bar{v}))\}$.

Now define the expanded language L_a with new constant symbols $\{a_1, a_2, ..., a_m\}$. Let $\bar{a}$ denote $(a_1, a_2, ..., a_m)$. We will begin with proving the following claim.

Claim: $T \cup \Gamma(\bar{a}) \vDash \varphi(\bar{a})$.
Sub-proof. Suppose not. Then there is a model $\mathcal{M} \vDash T \cup \Gamma(\bar{a}) \cup \neg\varphi(\bar{a})$. Let $\mathbb{A}$ be the substructure of $\mathcal{M}$ generated by $\bar{a}$.

Suppose $T \cup \mathrm{Diag}(\mathbb{A}) \cup \varphi(\bar{a})$ is not satisfiable. Then by the compactness theorem, there exist atomic (which of course is quantifier-free) formulae $\psi_1(\bar{a}), ..., \psi_n(\bar{a}) \in \mathrm{Diag}(\mathbb{A})$ such that $T \cup \{\psi_1(\bar{a}), ..., \psi_n(\bar{a})\} \cup \varphi(\bar{a})$ is not satisfiable. This means

$$T \vDash \varphi(\bar{a}) \rightarrow \bigvee_{i=1}^{i=n} \neg\psi_i(\bar{a}).$$

Note that the above is true under any interpretation of $\bar{a}$. So,

$$T \vDash \forall\bar{v}\left(\varphi(\bar{v}) \rightarrow \bigvee_{i=1}^{i=n} \neg\psi_i(\bar{v})\right).$$

Hence, $\bigvee_{i=1}^{i=n} \neg\psi_i(\bar{v}) \in \Gamma(\bar{v})$. So, we now know

$$\mathcal{M} \vDash \Gamma(\bar{a}) \vDash \bigvee_{i=1}^{i=n} \neg\psi_i(\bar{a}). \tag{1}$$

$$\mathbb{A} \leq \mathcal{M} \vDash \mathrm{Diag}(\mathbb{A}) \vDash \bigwedge_{i=1}^{i=n} \psi_i(\bar{a}). \tag{2}$$

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This is a contradiction to the fact $\mathcal{M}$ is a model. Which means, $T \cup \mathrm{Diag}(\mathbb{A}) \cup \varphi(\bar{a})$ is satisfiable.

So, there exists a model $\mathcal{N}$ of T such that $\mathcal{N} \vDash T \cup \mathrm{Diag}(\mathbb{A}) \cup \varphi(\bar{a})$. Because T is substructure complete and $\mathbb{A} \leq \mathcal{M} \vDash T$, we have that $\mathbb{A} \leq \mathcal{M} \vDash T \cup \mathrm{Diag}(\mathbb{A})$ decides $\varphi(\bar{a})$. But we know $\mathcal{M} \vDash \neg\varphi(\bar{a})$. Hence, $T \cup \mathrm{Diag}(\mathbb{A}) \vDash \neg\varphi(\bar{a})$.

Putting all these together, $\mathcal{N} \vDash T \cup \mathrm{Diag}(\mathbb{A}) \cup \varphi(\bar{a}) \vDash \neg\varphi(\bar{a}) \cup \varphi(\bar{a})$. This is a contradiction to the fact $\mathcal{N}$ is a model. Therefore, $T \cup \Gamma(\bar{a}) \vDash \varphi(\bar{a})$. $\square$

Now by the claim and compactness, there are $\psi_1(\bar{a}), ..., \psi_n(\bar{a}) \in \Gamma(\bar{a})$ such that

$$T \cup \{\psi_1(\bar{a}), ..., \psi_n(\bar{a})\} \vDash \varphi(\bar{a}), \text{ i.e.,}$$
$$T \vDash \bigwedge_{i=1}^{i=n} \psi_i(\bar{a}) \to \varphi(\bar{a}).$$

Note that the above is true under any interpretation of $\bar{a}$. So,

$$T \vDash \forall\bar{v} \left(\bigwedge_{i=1}^{i=n} \psi_i(\bar{v}) \to \varphi(\bar{v}) \right).$$

Since each $\psi_i(\bar{v})$ is in $\Gamma(\bar{v})$, $\displaystyle\bigwedge_{i=1}^{i=n} \psi_i(\bar{v})$ is quantifier-free and

$$T \vDash \forall\bar{v} \left(\bigwedge_{i=1}^{i=n} \psi_i(\bar{v}) \leftrightarrow \varphi(\bar{v}) \right).$$

Therefore, T has quantifier elimination.

Note: I ripped off this proof from the Thm 3.1.4. $(ii) \to (i)$ in David Marker's book. David Marker proved that, given models $\mathcal{M}$ and $\mathcal{N}$ of T and a L-structure $\mathbb{A} \subset \mathcal{M} \cap \mathcal{N}$, if $\mathcal{M} \vDash \varphi(\overline{a}) \Leftrightarrow \mathcal{N} \vDash \varphi(\overline{a})$ for all $\overline{a} \in A$, then T has quantifier elimination. The proof above is exactly the same as Marker's proof except for the last step of the claim; instead of using his antecedent, I used the substructure completeness of T to prove that T has quantifier elimination. $\square$