5. Let σ be a signature with one binary relation symbol and no other symbols. Show that there are exactly 2^κ nonisomorphic σ-structures of cardinality κ for each infinite κ.

Proof. Let R be the binary relation symbol in the signature σ. Consider σ-structures of cardinality κ, whose domain is κ (upto renaming elements). Since an interpretation of the binary relation R is a subset of $\kappa \times \kappa$, there will be at most $|\mathcal{P}(\kappa \times \kappa)| = 2^{\kappa^2} = 2^\kappa$ many interpretations. This implies there are at most 2^κ nonisomorphic σ-structures of cardinality κ for each infinite κ.

Now we aim to find the lower bound. First, we construct a σ-structure from each subset of κ. Then we show our σ-structures are different from each other, and so that there are at least $|\mathcal{P}(\kappa)| = 2^\kappa$ many distinct σ-structures. The key idea is to encode each subset I of κ by doubling the elements in I and copying the elements in $\kappa \setminus I$ while maintaining preorderedness and well-foundedness of the well-order $\leq$ in κ.

Let $\leq$ be the usual well-order on κ. Let π_i be the ith projection map. Given a subset $I \subset \kappa$, define $\mathbb{A}_I = (\{(I, x) \in \{I\} \times \kappa : x \in I\} \cup \{(\kappa \setminus I, x) \in \{\kappa \setminus I\} \times \kappa\}; \leq^{\mathbb{A}_I})$ where $(*, a) \leq^{\mathbb{A}_I} (*, b)$ iff $a \leq b$. Notice that $\leq^{\mathbb{A}_I}$ is a well-founded preorder and $\mathbb{A}_I$ is a σ-structure of cardinality κ. I claim $\mathbb{A}_I \not\cong \mathbb{A}_{I'}$ provided that $I \neq I'$. Suppose they are isomorphic. Say η is the isomorphism. Since η must preserve the well-founded preorder $\leq^{\mathbb{A}_I}$, η must be the identity map on the second component, i.e., $\pi_2 \circ \eta = \pi_2$. (This fact can be shown from the proof by contradiction). Now, WLOG, take $a \in I \setminus I'$. Then $(I, a), (\kappa \setminus I, a) \in \mathbb{A}_I$. So, $\eta(I, a) = \eta(\kappa \setminus I, a) = (\kappa \setminus I', a)$. Hence, η is not injective. This contradicts the fact η is an isomorphism. Therefore, subsets of κ generate distinct σ-structures. Which means, there are at least $|\mathcal{P}(\kappa)| = 2^\kappa$ nonisomorphic σ-structures of cardinality κ for each infinite κ.

Therefore, there will be exactly 2^κ nonisomorphic σ-structures of cardinality κ for each infinite κ. $\qquad\square$