When Is The Spectrum Of A Language With Countable Signature Scattered?

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Lemma 1. *The spectrum of a theory in a countable signature is scattered if and only if the space of complete theories is countable.*

Proof. Suppose that, for a language $\mathcal{L}$, the spectrum $\mathrm{Sp}(\mathcal{L})$ is not scattered. That is, suppose it contains a non-empty subset $X \subset \mathrm{Sp}(\mathcal{L})$ that has no isolated points. Then since the subspace X is a complete metric space with no isolated points, it has size at least as big as the continuum, c. It follows that the spectrum $\mathrm{Sp}(\mathcal{L})$ has uncountably many points.

Now suppose that the spectrum $\mathrm{Sp}(\mathcal{L})$ is a scattered topological space. Because it has a countable (or finite) signature, it must also be second-countable, that is it must have a countable basis. And because scattered spaces with countable basis are countable, the spectrum $\mathrm{Sp}(\mathcal{L})$ has countably many points. $\square$

Lemma 2. *If a language has finitely many unary relations and otherwise empty signature, then it has countably many complete theories.*

Proof. Suppose a language $\mathcal{L}$ has only finitely many unary relations and otherwise empty signature. We will show that the language $\mathcal{L}$ has countably many complete theories by counting models modulo elementary equivalence, a set in one-to-one correspondence with the set of complete theories. According to the Downward Lowenheim-Skolem Theorem, every language in a countable signature has a countable model, therefore it will be sufficient to count the $\mathcal{L}$-structures with countable universe. We will do this by showing that if two models have the same cardinality of elements satisfying, or not satisfying each subset of the unary relations in the signature, then they must be elementarily equivalent. Furthermore, since the number of such isomorphism classes of countable models is countable, this step will complete the proof.

We will write the relations in the language $\mathcal{L}$ as $\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_n$. Given a unary relation $\mathcal{R}_i$ in a model $\mathcal{M}$, we will write $(R_i)_M = \{x \in M : \mathcal{R}_i^{\mathcal{M}}(x)\}$. Also, for $1 \leq i \leq n$, we will write $R_{n+i} = \{x \in M : \neg\mathcal{R}_i^{\mathcal{M}}(x).\}$. Consider two models, $\mathcal{M}$ and $\mathcal{N}$ for which

$$|(R_{i_1})_M \cap ... \cap (R_{i_k})_M| = |(R_{i_1})_N \cap ... \cap (R_{i_k})_N|$$

for $1 \leq i_1 < i_2 < ... < i_k \leq 2n$.

Now, because the cardinality of the elements satisfying or not satisfying any subset of the relations is the same in both models, there must exist some bijection $\phi : M \to N$ such that $\mathcal{R}_i^{\mathcal{M}}(x)$ if and only if $\mathcal{R}_i^{\mathcal{N}}(\phi(x))$. Now the map ϕ is an isomorphism between the $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, which proves that $Th(\mathcal{M}) = Th(\mathcal{N})$. This gives us countably many isomorphism types of countable models and completes the proof. □

Lemma 3. *A language with at least one unary function or at least one binary relation has uncountably many complete theories.*

Proof. Since a binary relation can model a unary function, it is sufficient to prove the unary function part of the claim. To do this, we will show that a language $\mathcal{L}$ with one unary function $\mathcal{F}$ has uncountably many models that are pairwise elementarily inequivalent. Suppose the signature of the language $\mathcal{L}$ contains some unary function $\mathcal{F}$. For the interpretation of $\mathcal{F}$ in the model $\mathcal{F}$ we will write $\mathcal{F}^{\mathcal{M}}$ and for its nth iterate we will write $(\mathcal{F}^{\mathcal{M}})^n$.

There are uncountably many subsets of the natural numbers. For each subset $S \subset \mathbb{N}$, let $\mathbb{N}_S$ be the $\mathcal{L}$-structure whose universe is $\mathbb{N}$ and for which the function $\mathcal{F}^{\mathcal{M}}$ is a permutation with orbits have size s for each $s \in S$ (and no other sizes). Any two distinct subsets $S, T \subset \mathbb{N}$ will determine elementarily inequivalent $\mathcal{L}$-structures, because if $t \in T$ and $t \notin S$ then the first order statement

$$\exists x[((\mathcal{F}^{\mathcal{M}})^t(x) = x) \wedge (\bigwedge_{i=1,2,\ldots,t-1} (\mathcal{F}^{\mathcal{M}})^i(x) \neq x)]$$

will be true of $\mathbb{N}_T$ and will not be true of $\mathbb{N}_S$, which proves that they have distinct theories. Therefore the language $\mathcal{L}$ has uncountably many complete theories. □

Lemma 4. *A language with infinite signature has uncountably many complete theories.*

Proof. Note that a relation or function of any order (except a 0-ary relation) can model a constant. Therefore it is sufficient to prove the cases in which their are either infinitely many constants or infinitely many 0-ary relations in the signature. In a language with infinitely many constants, there are uncountably many isomorphism types of two element structures, no two are which are elementarily equivalent. These structures each has its own complete theory, leading to uncountably many complete theories. In the case of infinitely many 0-ary relations, it is possible to take the empty universe and assign true or false to the relations in uncountably many ways, since each assignment of true or false corresponds to the subset of the relations to which we assign the value true. Any two of these assignments have elementarily inequivalent theories, which gives us uncountably many complete theories. □

Theorem 5. *A language in a countable signature is scattered if and only if it*

1. *it has no functions that are 1-ary or higher,*

2. *it has no relations that are 2-ary or higher,*

3. *it has a finite signature.*

Proof. Recall from Lemma 1 that a language in a countable signature is scattered if and only if it has countably many complete theories.

Suppose that the language $\mathcal{L}$ in countable signature has countably many complete theories. By Lemma 3, this cannot happen if the signature contains any functions of degree at least 1 or relations of degree at least 2. Also, by Lemma 4, any language with countably many complete theories must have a finite signature.

Now suppose that the conditions stated above hold. Then the signature has just finitely many constants, finitely many functions and finitely many 0-ary or unary relations. Since all of the above can be modeled by unary relations, and by Lemma 2 any theory with only finitely many unary relations in its signature has only countably many complete theories, such a theory can have only countably many complete theories. $\square$