MATH 6000 - Assignment 1

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3. Show that the theory of finite groups is different than the theory of all groups.

Proof. Take $\mathcal{L} = \{\cdot, e\}$ to be the language of groups. If there is some $\mathcal{L}$-sentence ϕ that is true in every finite group and false in at least one group, then ϕ is in the theory of finite groups and not in the theory of all groups. Consider

$$\phi = [(\forall x \forall y(x \cdot x = y \cdot y \rightarrow x = y)) \rightarrow (\forall x \exists y(x = y \cdot y))].$$

The sentence ϕ essentially states that if the squaring or doubling function is an injection, then it is a surjection. The sentence ϕ must be true in every finite group since any function from a finite set to itself is an injection if, and only if, it is a surjection. Now examine the group $< \mathbb{Z}; +, 0 >$. Under ordinary addition the double of any integer is unique, thus satisfying the antecedent of ϕ. However, only the even integers will be the output of the doubling function meaning that it is not a surjection. Therefore, the sentence ϕ cannot be in the theory of all groups but it is in the theory of finite groups. This allows us to conclude that the theory of all groups is indeed distinct from the theory of finite groups. $\blacksquare$