Raymond Baker
Thomas Magnuson
Mateo Muro

12. Let $\mathcal{K}$ be a class of $\mathcal{L}$ structures. Show that the ultraproduct of ultraproducts of members of $\mathcal{K}$ is isomorphic to an ultrapoduct of members of $\mathcal{K}$.

Proof. Let K' be an ultraproduct of ultraproducts of members of $\mathcal{K}$. That is, for infinite index sets $I, \{J_i\}_{i\in I}$ and ultrafilters $V, \{U_i\}_{i\in I}$,

$$K' = \Pi_{i\in I}[(\Pi_{j\in J_i} K_{i,j})/U_i]/V$$

We define a new index set $M = \bigsqcup_{i\in I} J_i$, the disjoint union over the J_i's. We also define an ultrafilter W on M. For subset $P \subseteq M$, we say $P \in W$ if and only if $\{i \in I \mid (P \cap J_i) \in U_i\} \in V$. That is, if the set indices where P is large is large, then $P \in W$.
We show this is a filter. Suppose we have $P \in W$ and $P \subseteq R$. Then we know that $\{i \in I \mid (P \cap J_i) \in U_i\} \in V$ and $\{i \in I \mid (P \cap J_i) \in U_i\} \subseteq \{i \in I \mid (R \cap J_i) \in U_i\}$. Then since V is itself a filter, we have $\{i \in I \mid (R \cap J_i) \in U_i\} \in V$ and $R \in W$. Now suppose that $P, R \in W$. Then we have $\{i \in I \mid (P \cap J_i) \in U_i\} \in V$ and $\{i \in I \mid (R \cap J_i) \in U_i\} \in V$. Since V is a filter, $\{i \in I \mid (P \cap R \cap J_i) \in U_i\} = \{i \in I \mid (P \cap J_i) \in U_i\} \cap \{i \in I \mid (R \cap J_i) \in U_i\} \in V$ and we have $P \cap R \in V$.
Now we need to show that W is an ultrafilter. Let $P \subseteq M$ and consider $P_* = \{i \in I \mid (P \cap J_i) \in U_i\}$. Because V is an ultrafilter, either P_* or P_*^c is in V. If $P_* \in V$, we have $P \in W$ and are done. Otherwise, P^c will be in W because $P_*^c = \{i \in I \mid (P \cap J_i) \in U_i\}^c = \{i \in I \mid (P^c \cap J_i) \in U_i\} \in V$, where the second inequality comes from the fact that each U_i is an ultrafilter.
Now we want to show that our new ultraproduct

$$(\Pi_{m\in M} K_m)/W$$

is isomorphic to the ultraproduct K'. We do so by comparing the kernels of two quotient maps. $q_1 : \Pi_{m\in M} K_m \to (\Pi_{m\in M} K_m)/W$ and $q_2 : \Pi_{i\in I}[(\Pi_{j\in J_i} K_{i,j}) \to K'$. Notice that $q_1(a) = q_1(b)$ if and only if $\{m \in M \mid a_m = b_m\} \in W$. To reiterate, that happens if and only if $\{i \in I \mid \{j \in J_i \mid a_j = b_j\} \in U_i\} \in V$. These are also the same elements that $q_2(a) = q_2(b)$. Hence, the two quotient maps have the same kernel. Since the quotient of a quotient is a quotient, we have that the two target structures are isomorphic. □