MODEL THEORY HOMEWORK 3

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Problem (2)**.** If $\mathbb{A}$ and $\mathbb{B}$ are elementarily equivalent L-structures, then there exists an L-structure into which $\mathbb{A}$ and $\mathbb{B}$ can be elementarily embedded.

Proof. We may safely assume that the underlying sets A of $\mathbb{A}$ and B of $\mathbb{B}$ are disjoint (if not, simply take isomorphic structures that re-label the underlying sets so that they are). Let $L(A)$ denote the language L expanded by constants from A; similarly define $L(B)$ and $L(A \cup B)$. Note $\mathrm{Th}(\mathbb{A}_A)$ is a complete $L(A)$-theory and also an $L(A \cup B)$-theory. Likewise $\mathrm{Th}(\mathbb{B}_B)$ is a complete $L(B)$-theory and an $L(A \cup B)$-theory.

We claim that $T = \mathrm{Th}(\mathbb{A}_A) \cup \mathrm{Th}(\mathbb{B}_B)$ is a satisfiable $L(A \cup B)$-theory. It suffices by compactness to show that T is finitely satisfiable. Indeed, take a finite subset $S \subset T$. Certainly if S involves no constants from $A \cup B$, then $S \subset \mathrm{Th}(\mathbb{A}) = \mathrm{Th}(\mathbb{B})$ and both $\mathbb{A}$ and $\mathbb{B}$ are models of S. Otherwise, suppose $\overline{a}$ is the finite set of n constants from A mentioned in S (and finitely many constants from B are possibly mentioned as well). Note that $\mathbb{B}_B$ models every sentence in $S \cap \mathrm{Th}(\mathbb{B}_B)$. Denote the remaining sentences as $S_A = S \cap \mathrm{Th}(\mathbb{A}_A)$. For any sentence φ in S_A, φ is actually an $L(A)$-sentence, and there is an n-ary L-formula $\varphi'(\overline{x})$ obtained by replacing occurrences of $\overline{a}$ with the free variable(s) $\overline{x}$. Since $\mathbb{A}_A$ models φ, $\mathbb{A}$ itself models $\exists \overline{x}\ \varphi'(\overline{x})$, which is a plain L-sentence. Moreover, we can say $\mathbb{A} \models \exists \overline{x}\ \bigwedge_{\varphi \in S_A} \varphi'(\overline{x})$ which is another L-sentence since there are finitely many sentences in S_A. But then $\mathbb{B}$ models the same sentence, and there is a tuple $\overline{b} \in B^n$ serving as witness. Then we may make $\mathbb{B}_B$ into an $L(A \cup B)$-structure by interpreting $\overline{a}^{\mathbb{B}_B} = \overline{b}$ (and the rest of A in an arbitrary way). Each $L(A \cup B)$-sentence in S will hold in this structure by construction, so S is satisfiable.

Let $\mathbb{C}$ be a model of T. $\mathbb{C}$ contains an isomorphic copy of $\mathbb{A}_A$ and $\mathbb{B}_B$; explicitly we may think of $\mathbb{A} = \left\{ a^{\mathbb{C}} : a \in A \right\} \subset \mathbb{C}$ and likewise for $\mathbb{B}$. Now we verify that this copy of $\mathbb{A}$ is an elementary substructure by applying the Tarski-Vaught criterion. Indeed, take an L-formula $\psi(x, \overline{y})$ and $\overline{a} \in A$, and suppose $\mathbb{C}$ models $\sigma = \exists x\ \psi(x, \overline{a})$. But σ is an $L(A)$-sentence and $\mathbb{C}$ models the complete theory $\mathrm{Th}(\mathbb{A}_A)$, so if σ were *not* in $\mathrm{Th}(\mathbb{A}_A)$, then $\neg\sigma$ would be, and $\mathbb{C} \models \sigma \wedge \neg\sigma$ which cannot be the case. So $\mathbb{A}_A \models \sigma$ and we have an $a_0 \in A$ with $\mathbb{A} \models \psi(a_0, \overline{a})$ as desired. So $\mathbb{A}$ and, by the same argument $\mathbb{B}$, are elementary substructures of $\mathbb{C}$. $\square$