MATH 6000 - Assignment 1

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2. Show that the class of simple groups is not elementary.

Proof. Let $\mathcal{L} = \{\cdot, e\}$ be the language of groups, and for contradiction suppose the class of simple groups, $\mathcal{K}$, is elementary. Then there is an $\mathcal{L}$-theory T such that

$$\mathcal{K} = \{G : G \models T\}.$$

But then the class of simple abelian groups, $\mathcal{K}_{\text{ab}}$, is also elementary, because of the following argument. Take

$$T' := T \cup \{\forall x \forall y (x \cdot y = y \cdot x)\}$$

Then if $G \in \mathcal{K}_{\text{ab}}$ we have

$$G \models T$$

and

$$G \models \{\forall x \forall y (x \cdot y = y \cdot x)\}$$

and so

$$G \models T'.$$

On the other hand, if $G \models T'$ then $G \models T$ and $G \models \{\forall x \forall y (x \cdot y = y \cdot x)\}$. From the assumption that simple groups are an elementary class we get that $G \in \mathcal{K}$ and G is abelian, from which it follows that $G \in \mathcal{K}_{\text{ab}}$. Therefore

$$\mathcal{K}_{\text{ab}} = \{G : G \models T'\}.$$

.

Since every nontrivial abelian simple group is isomorphic to $\mathbf{Z}/p\mathbf{Z}$ for some prime p, we claim that every finite subset of

$$T'' := T' \cup \left\{ \forall x \left(\left(\bigwedge_{1<i\leq n} \neg(x = \underbrace{x \cdot \dots \cdot x}_{i \text{ times}}) \right) \vee (x = e) \right) : n \in \mathbf{Z}_{>1} \right\}$$

is satisfiable in $\mathcal{K}_{\mathrm{ab}}$. If $\widetilde{T}$ is a finite subset of T'', then $\widetilde{T}$ contains only finitely many statements asserting that a group does not have elements of a given order. But because there are infinitely many primes, it is always possible to select a prime p larger than all those mentioned in $\widetilde{T}$. In fact, since every nonidentity element in group $\mathbb{Z}/p\mathbb{Z}$ has order p, we have that $\mathbf{Z}/p\mathbf{Z} \models \widetilde{T}$. Therefore every finite sub-theory $\widetilde{T}$ is satisfiable and by the Compactness Theorem the full theory T'' is also satisfiable. However, if $G \models T''$, then G is infinite and abelian, so cannot be simple. This is a contradiction, so $\mathcal{K}$ is not elementary. ■