Quantifiers!

The quantifiers $\forall$ (forall) and $\exists$ (there exists) are modifiers which are used to express how much or how many.

Some attribute to Abraham Lincoln the quote

> *You can fool* <u>*all*</u> *the people* <u>*some*</u> *of the time and* <u>*some*</u> *of the people* <u>*all*</u> *the time, but you cannot fool* <u>*all*</u> *the people* <u>*all*</u> *the time.*

We can formalize this statement using a predicate $F(p, t)$ expressing "Person p can be fooled at time t". Here p is a variable allowed to range over a set P of people and t is a variable allowed to range over a set T of times. Now the statement reads

$$((\exists t)(\forall p)F(p,t)) \wedge ((\exists p)(\forall t)F(p,t)) \wedge \neg((\forall t)(\forall p)F(p,t)).$$

To see the effect of quantification on a predicate, let $P = \{\text{John}, \text{Paul}, \text{George}, \text{Ringo}\}$, let $T = \{8\text{am}, 11\text{am}, 2\text{pm}, 5\text{pm}\}$, and assume the fooling predicate, $F(p,t)$, is given by the table:

$F(p,t)$	8	11	2	5
John	0	0	1	0
Paul	1	1	1	1
George	0	1	1	0
Ringo	0	0	1	1

Now let's examine the tables for each of $(\forall p)F(p,t)$, $(\exists p)F(p,t)$, $(\forall t)F(p,t)$, $(\exists t)F(p,t)$:

$(\forall p)F$	8	11	2	5
J	0	0	1	0
P	0	0	1	0
G	0	0	1	0
R	0	0	1	0

$(\exists p)F$	8	11	2	5
J	1	1	1	1
P	1	1	1	1
G	1	1	1	1
R	1	1	1	1

$(\forall t)F$	8	11	2	5
J	0	0	0	0
P	1	1	1	1
G	0	0	0	0
R	0	0	0	0

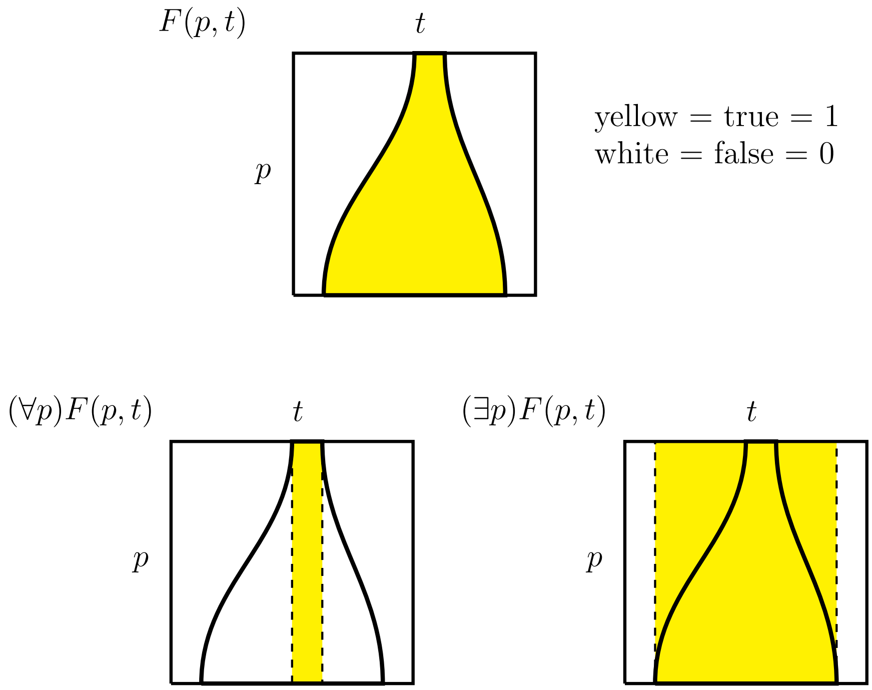
$(\exists t)F$	8	11	2	5
J	1	1	1	1
P	1	1	1	1
G	1	1	1	1
R	1	1	1	1

Finally, let's examine the tables for $(\exists t)(\forall p)F(p,t)$, $(\exists p)(\forall t)F(p,t)$, and $(\forall t)(\forall p)F(p,t)$:

$(\exists t)(\forall p)F$	8	11	2	5
John	1	1	1	1
Paul	1	1	1	1
George	1	1	1	1
Ringo	1	1	1	1

$(\exists p)(\forall t)F$	8	11	2	5
John	1	1	1	1
Paul	1	1	1	1
George	1	1	1	1
Ringo	1	1	1	1

$(\forall t)(\forall p)F$	8	11	2	5
John	0	0	0	0
Paul	0	0	0	0
George	0	0	0	0
Ringo	0	0	0	0



The effects of quantifying the predicate $F(p,t)$ with respect to the variable p are:

(1) The resulting predicate is independent of the quantified variable, p.
(2) This is expressed through the "cylindrification" in the p-direction of the support of the predicate.
(3) If the quantifier is $\forall$, then you use internal cylindrification. If the quantifier is $\exists$, then you use external cylindrification.

In particular,

(4) If a predicate is quantified a second time with respect to the same variable (as in $(\forall p)(\exists p)F(p,t))$, the second (= leftmost = outermost) quantifer causes no further change in meaning.
(5) If a predicate is quantified with respect to each of its variables, then the resulting predicate is either True or False. That is, its table is either filled with 1's only or it is filled with 0's only.

Some terminology:

- Scope of a quantifier.
- Free variable. Bound variable.

Handling Quantifiers in Practice.

On the rest of this handout, I will discuss the truth of statements written in *prenex form*, which means statements where the quantifiers appear in the front.

Examples:

(1) Are the statements $(\exists t)(\forall p)F(p,t)$, $(\exists p)(\forall t)F(p,t)$, and $(\forall t)(\forall p)F(p,t)$, where F is the fooling predicate, true in the structure $\langle \{\mathrm{J},\mathrm{P},\mathrm{G},\mathrm{R}\}, \{8,11,2,5\}; F(p,t)\rangle$?
(2) Is $(\forall x)(x \geq 0)$ true in $\mathbb{N}$? in $\mathbb{R}$?
(3) Is $(\forall x)(\exists y)(x = y)$ true in $\mathbb{N}$? in $\mathbb{R}$?
(4) Is $(\forall x)(\exists y)(x > y)$ true in $\mathbb{N}$? in $\mathbb{R}$?
(5) Is $(\forall a)(\exists b)(\forall c)(\exists d)(a^2 + b^2 = c^2 + d^2)$ true in $\mathbb{R}$? in $\mathbb{C}$?
(6) Is $(\forall a)(\exists b)(\forall c)(\exists d)(\forall e)(\exists f)(\forall g)(a < b < c < d < e < f < g)$ true in $\mathbb{N}$? in $\mathbb{R}$?

Quantifier Games.

- Two players $\forall$ (= Abelard) and $\exists$ (= Eloise).
- Order of play: is the order of the quantifiers in the quantifier prefix.
- Game template: is the "body" of the formula.
- A "strategy" for Player P is a function that, at Play k, computes a move for Player P from an examination of all earlier moves by either player.
- A sentence is true iff Eloise has a winning strategy. A sentence is false iff Abelard has a winning strategy. One of these must occur.

Example 1. Is $(\forall x)(\exists y)(x = y)$ true in $\mathbb{N}$? in $\mathbb{R}$?

I claim that the sentence is true in both $\mathbb{N}$ and $\mathbb{R}$. The same strategy is a winning strategy for $\exists$ in both structures. It is

Play 1 ($\forall$): Let $\forall$ choose some value for x.
Play 2 ($\exists$): Choose y equal to the value chosen for x in Play 1.

Example 2. Is $(\forall x)(x \geq 0)$ true in $\mathbb{N}$? in $\mathbb{R}$?

I claim that the sentence is true in $\mathbb{N}$, but false in $\mathbb{R}$.

To establish truth in $\mathbb{N}$, I give a winning strategy for $\exists$: The strategy is "Do nothing". More precisely, it is the strategy

Play 1 ($\forall$): Let $\forall$ choose some value for x. ($\exists$ cannot control this choice.)

This is the only strategy, winning or not, for $\exists$. Why is it a winning strategy?

To establish falsity in $\mathbb{R}$, I give a winning strategy for $\forall$:

Play 1 ($\forall$): Choose $x = -1$.

Why is it a winning strategy?

Practice!

Give winning strategies for the appropriate quantifier.

(1) $(\exists t)(\forall p)F(p,t)$ in the structure $\langle\{\mathrm{J},\mathrm{P},\mathrm{G},\mathrm{R}\},\{8,11,2,5\};F(p,t)\rangle$.

Play 1 ($\exists$):
Play 2 ($\forall$):

(2) $(\exists p)(\forall t)F(p,t)$ in the structure $\langle\{\mathrm{J},\mathrm{P},\mathrm{G},\mathrm{R}\},\{8,11,2,5\};F(p,t)\rangle$.

(3) $(\forall p)(\forall t)F(p,t)$ in the structure $\langle\{\mathrm{J},\mathrm{P},\mathrm{G},\mathrm{R}\},\{8,11,2,5\};F(p,t)\rangle$.

(4) Is $(\forall x)(\exists y)(x > y)$ true in $\mathbb{N}$? in $\mathbb{R}$?

(5) Is $(\forall a)(\exists b)(\forall c)(\exists d)(a^2 + b^2 = c^2 + d^2)$ true in $\mathbb{R}$? in $\mathbb{C}$?

(6) Is $(\forall a)(\exists b)(\forall c)(\exists d)(\forall e)(\exists f)(\forall g)(a < b < c < d < e < f < g)$ true in $\mathbb{N}$? in $\mathbb{R}$?