

## Theorems and their proofs!

- A **Theorem** is a declarative statement that has a *proof*.
- A **Proof** of a theorem  $T$  is a finite sequence of statements  $S_1, \dots, S_k = T$  such that
  - each statement follows from earlier statements via an accepted *law of deduction*.
  - the last statement in the proof is  $T$ .
- A **Law of Deduction** is rule for drawing a *conclusion* from a set of *premises*.

### Examples of laws of deduction.

- (1) (Modus Ponens, or “the mode that affirms”) Written  $\frac{(P \rightarrow Q), P}{Q}$ . It means that if, within a proof, you see  
 $S_1, S_1, \dots, P, \dots, (P \rightarrow Q), \dots, S_m,$  or  $S_1, S_1, \dots, (P \rightarrow Q), \dots, P, \dots, S_m,$   
then you can continue the proof  
 $S_1, S_1, \dots, P, \dots, (P \rightarrow Q), \dots, S_m, Q$  or  $S_1, S_1, \dots, (P \rightarrow Q), \dots, P, \dots, S_m, Q.$
- (2) (Modus Tollens, or “the mode that denies”) Written  $\frac{(P \rightarrow Q), \neg Q}{\neg P}$ .
- (3) (Axioms) Written  $\frac{}{A}$ .

### Topics for discussion.

- (1) How do we select axioms? rules of deduction?
- (2) What is the role of the hypotheses of a theorem?
- (3) What is the difference between formal proof and informal proof?
- (4) Are all provable statements true? Are all true statements provable? (Soundness, completeness, decidability, Gödel’s Completeness Theorem.)
- (5) What are some common proof strategies?
  - (a) Direct proof.
  - (b) Proof by contradiction.
  - (c) Proof of the contrapositive.
  - (d) Read “Tactics for logical reasoning”, subsection 4.8.2.
- (6) How can we use propositional tautologies when constructing proofs?

**Practice:** Prove the theorems directly, by contraposition, and by contradiction (3 proofs each):

**Thm.** If  $x > 0$ , then  $x + 1 > 0$ .

**Thm.** If  $0 < x < 1$ , then  $x^2 < x$ .

**Thm.** If  $n$  is even and  $n = k^2$ , then  $k$  is even.