Practice with Logic and Counting! (Some hints!)

(1) One language for ordered sets has $\leq$ as its only nonlogical symbol. In this language, write a formula $\varphi(x)$ expressing that "x is not the largest element and not the smallest element."

$$\underbrace{\neg((\forall y)(x \leq y))}_{x \text{ is not smallest}} \wedge \underbrace{\neg((\forall z)(z \leq x))}_{x \text{ is not largest}}$$

(2) Write a formal sentence that expresses Fermat's Last Theorem in a language for number theory whose nonlogical symbols are $0, +, \cdot, \hat{}, <$. (Fermat's Last Theorem is the statement that if x, y, z, n are nonzero natural numbers and n is at least 3, then $x\hat{}n + y\hat{}n = z\hat{}n$ does not hold.)

A first guess might be

$$\forall x\, \forall y\, \forall z\, \forall n\, ((x\hat{}n + y\hat{}n = z\hat{}n) \to ((x = 0) \vee (y = 0) \vee (z = 0) \vee (n < 3))),$$

but unfortunately this is not a statement in the given language. The problem is that we do not have a symbol in the language for the number 3. Therefore we have to define 3 from the symbols we have. One way to do this is to first define the number 1, then construct 3 from 1.

Let $1(x) =$ "$\forall y\ (x \cdot y = y)$." When applied to a natural number x the formula $1(x)$ is true only when $x = 1$. Now define $2(x) = \exists y\ (1(y) \wedge (x = y + y))$, $3(x) = \exists y\ \exists z\ (1(y) \wedge 2(z) \wedge (x = y + z))$, etc. Now, the formula we seek is

$$\forall x\, \forall y\, \forall z\, \forall n\, ((x\hat{}n + y\hat{}n = z\hat{}n) \to ((x = 0) \vee (y = 0) \vee (z = 0) \vee (\exists w\ ((3(w) \wedge (n < w))))))$$

(Note: There are other ways to define the number 3 in this language, such as "$0\hat{}0 + 0\hat{}0 + 0\hat{}0$".)

(3) If $(a_n)_{n\in\mathbb{N}} = (a_0, a_1, a_2, \ldots)$ is a sequence of real numbers, then we use the abbreviation "$\lim\limits_{n\to\infty} a_n = L$" to mean

$$(\forall k > 0)(\exists N)(\forall n)\left((n > N) \to \left(|a_n - L| < \frac{1}{k}\right)\right).$$

(a) If $(a_n) = (1, 1, 1, \ldots)$, show that $\lim_{n\to\infty} a_n = 1$.

(b) If $(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$, show that $\lim_{n\to\infty} a_n = 0$.

(c) If $(a_n) = (1, 2, 3, \ldots)$, show that $\lim_{n\to\infty} a_n \neq 0$.

(4)

(a) Show that if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then $\lim_{n\to\infty}(a_n + b_n) = A + B$.

(b) Expand the following into a formal sentence and then draw the formula tree for that sentence.

$$(((\lim_{n\to\infty} a_n = A) \quad \text{and} \quad (\lim_{n\to\infty} b_n = B)) \quad \text{implies} \quad (\lim_{n\to\infty}(a_n + b_n) = A + B)).$$

(5) Rewrite the following sentences so that they are in prenex form, and the bodies of the sentences are in disjunctive normal form.

(a) $(\forall x)(x = x) \to (\exists x)(x = x)$.

First, prenex only: $(\exists x)(\exists y)((x = x) \to (y = y))$. Next, recall that $P \to Q \equiv (\neg P) \vee Q$ has DNF equal to

$$((\neg P) \wedge (\neg Q)) \vee ((\neg P) \wedge Q) \vee (P \wedge Q).$$

Hence, the full answer is

$$(\exists x)(\exists y)(((\neg(x = x)) \wedge (\neg(y = y))) \vee ((\neg(x = x)) \wedge (y = y)) \vee ((x = x) \wedge (y = y)))$$

(b) $(\forall x)(\forall y)((x < y) \to (\exists z)(x < z < y))$.
(Here $x < z < y$ is an abbreviation for $(x < z) \wedge (z < y)$.)

(6) If possible, give an example of a propositional formula (or "truth function") that uses only the variables P and Q and only the logical connectives $\wedge, \vee, \neg, \rightarrow$ and $\leftrightarrow$, which satisfies the following three properties.

(a) The formula is a tautology,

(b) If you replace every instance of P with $\neg P$ in the formula it remains a tautology, but

(c) If you replace every instance of Q with $\neg Q$ in the formula it changes to a contradiction.

It is not possible. If the formula is a tautology, then it remains one after any variable substitution (such as $Q \mapsto \neg Q$).

(7) Write an informal explanation of what it means for a proof system to have each of the following characteristics.

(a) Soundness.

(b) Completeness.

(c) Decidability.

(8) Explain the difference between truth and provability.

(9) How many 7-digit numbers have all of these properties?

- All digits are distinct,
- the leading digit is not 0,
- no two consecutive digits are even, and no two consecutive digits are odd.

The number of such numbers that start with an odd digit is $5 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2$. The number that start with a nonzero even digit is $4 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2$. Hence the answer is $5 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 + 4 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 = 12960$.

(10) Thirty dots are evenly spaced on the circumference of a circle. How many ways can we choose a subset of these dots if we must pick at least three dots and we are not allowed to choose exactly the set of vertices of a regular polygon?

$$\left(2^{30} - \binom{30}{0} - \binom{30}{1} - \binom{30}{2}\right) - \left(\frac{1}{3}\binom{30}{1} + \frac{1}{5}\binom{30}{1} + \frac{1}{6}\binom{30}{1} + \frac{1}{10}\binom{30}{1} + \frac{1}{15}\binom{30}{1} + \frac{1}{30}\binom{30}{1}\right)$$

(11)

(a) How many binary relations on the set $n = \{0, 1, \ldots, n-1\}$ are there?

2^{n^2}

(b) How many binary relations on n are reflexive?

2^{n^2-n}

(c) How many binary relations on n are reflexive and symmetric?

$2^{(n^2-n)/2}$

(d) Explain why there are B_n binary relations on n that are reflexive, symmetric, and transitive.

B_n is the number of partitions of n, which equals the number of equivalence relations on n.

(12) These problems are about seating people at a round table. Two seating arrangements are considered the same if they differ by a rotation. (So, for example, the arrangement $ABCDEF$ is the same as $BCDEFA$.)

(a) How many ways are there to sit 3 couples at a round table?

$6!/6 = 5! = 120$.

(b) What if couples must sit together? Treat each couple, say AB, as a large person who may sit in one of two ways AB or BA.
Seating 3 large people can be accomplished in $3!/3 = 2! = 2$ ways. But each large person can be seated in 2 ways, so the total is $2 \cdot 2^3 = 16$.

(c) What if couples are not allowed to sit together? Let Property i be the property that Couple i sits together, $i = 1, 2, 3$. We want

$$\begin{aligned} N_=(\emptyset) &= N_\geq(\emptyset) - \binom{3}{1} N_\geq(P_i) + \binom{3}{2} N_\geq(P_i, P_j) - \binom{3}{0} N_\geq(P_1, P_2, P_j) \\ &= 5! - 3 \cdot 4! \cdot 2 + 3 \cdot 3! \cdot 2^2 - 2! \cdot 2^3 \\ &= 32. \end{aligned}$$

(13) Imagine the following two events concerning ordinary 6-sided dice.

(a) You roll a single die twice and the sum of values is 7.

(b) You roll a single die four times and the sum of values is 14.

Is the first event more probable, equally probable, or less probable than the second event?

The first event is more probable.

There are $6^2 = 36$ ways to roll a die twice, and $C(7-1, 2-1) = 6$ of these result in a sum of 7. Hence the probability of the first event is $6/36 = 1/6 = .16666\ldots$.
There are 6^4 ways to roll a die four times, and we want to count the number of ways in which the sum is 14. This is the problem of distributing 14 identical balls (the sum) to four distinct boxes (the rolls) in such a way that every box gets at least 1 and not more than 6. This involves inclusion and exclusion.

We count distributions of balls to boxes where each box gets at least 1 ball. Let P_i be the property of a distribution that box i gets more than 6 balls. The number $N_=(\emptyset)$ is

$$C(14-1,4-1)-\binom{4}{1}C(8-1,4-1)+\underbrace{\binom{4}{2}C(2-1,4-1)}_{\text{already }=0}-\text{ETC}=146.$$

Hence the probability of the second event is $146/6^4 = .1126543\ldots$.