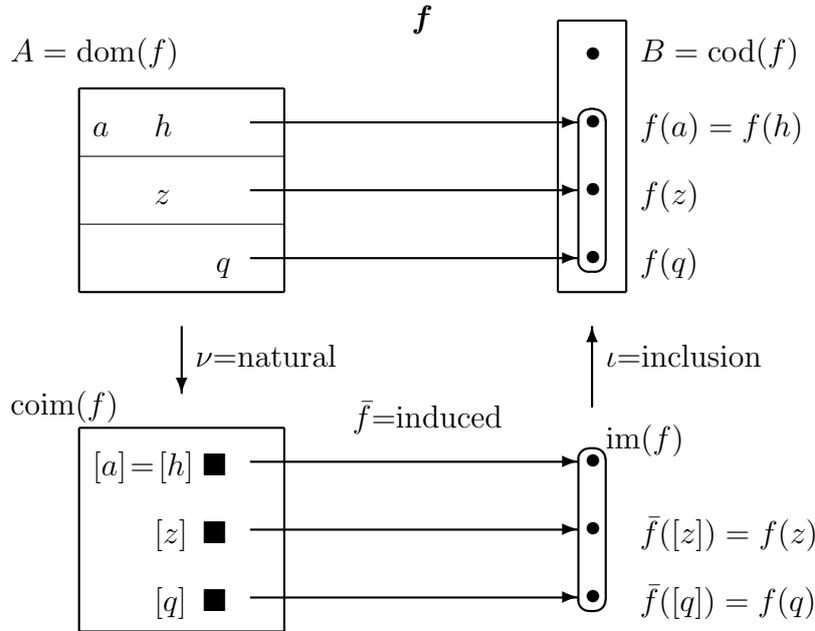


## Terminology for functions.

Let  $A$  and  $B$  be sets and let  $f: A \rightarrow B$  be a function from  $A$  to  $B$ . There are sets and functions related to  $A, B$  and  $f$  that have special names.



- (1) The *image* of  $f$  is  $\text{im}(f) = f[A] = \{b \in B : \exists a \in A(f(a) = b)\}$ . The image of a subset  $U \subseteq A$  is  $f[U] = \{b \in B : \exists u \in U(f(u) = b)\}$ .
- (2) The *preimage* or *inverse image* of a subset  $V \subseteq B$  is  $f^{-1}[V] = \{a \in A : f(a) \in V\}$ .
- (3) The preimage of a singleton  $\{b\}$  is written  $f^{-1}(b)$  and sometimes called the *fiber* of  $f$  over  $b$ . The fiber containing the element  $a$  is sometimes written  $[a]$ .
- (4) The *coimage* of  $f$  is the set  $\text{coim}(f) = \{f^{-1}(b) : b \in \text{im}(f)\}$  of all nonempty fibers.
- (5) The *natural map* is  $\nu: A \rightarrow \text{coim}(f): a \mapsto [a]$ .
- (6) The *inclusion map* is  $\iota: \text{im}(f) \rightarrow B: b \mapsto b$ .
- (7) The *induced map* is  $\bar{f}: \text{coim}(f) \rightarrow \text{im}(f): [a] \mapsto f(a)$ .

Some facts:

- (1) The natural map is *surjective*.
- (2) The inclusion map is *injective*.
- (3) The induced map is *bijective*.
- (4)  $f = \iota \circ \bar{f} \circ \nu$ . (This is the *canonical factorization* of  $f$ .)

**Practice problems.**

- (1) Draw a figure like the previous one illustrating  $f: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto x^2$ . Identify all the “named” sets and functions.
  
- (2) Repeat the previous exercise for the function  $g: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \mapsto x + y$ .
  
- (3) Repeat for the *identity function*  $\text{id}: A \rightarrow A: a \rightarrow a$ .
  
- (4) Repeat for the *second coordinate projection*  $\pi: X \times Y \rightarrow Y: (x, y) \rightarrow y$ .
  
- (5) Show that
  - (a) the composition of two injective functions is injective,
  - (b) the composition of two surjective functions is surjective, and
  - (c) the composition of two bijective functions is bijective.
  
- (6) Show that injective functions are *left cancellable*: if  $f$  is injective, then  $f \circ g = f \circ h$  implies  $g = h$ .
  
- (7) Show that surjective functions are *right cancellable*: if  $f$  is surjective, then  $g \circ f = h \circ f$  implies  $g = h$ .
  
- (8) Show that if  $f: A \rightarrow B$  is a function, then  $f^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$  is also a function. Show that  $f$  is injective iff  $f^{-1}$  is surjective, and  $f$  is surjective iff  $f^{-1}$  is injective.