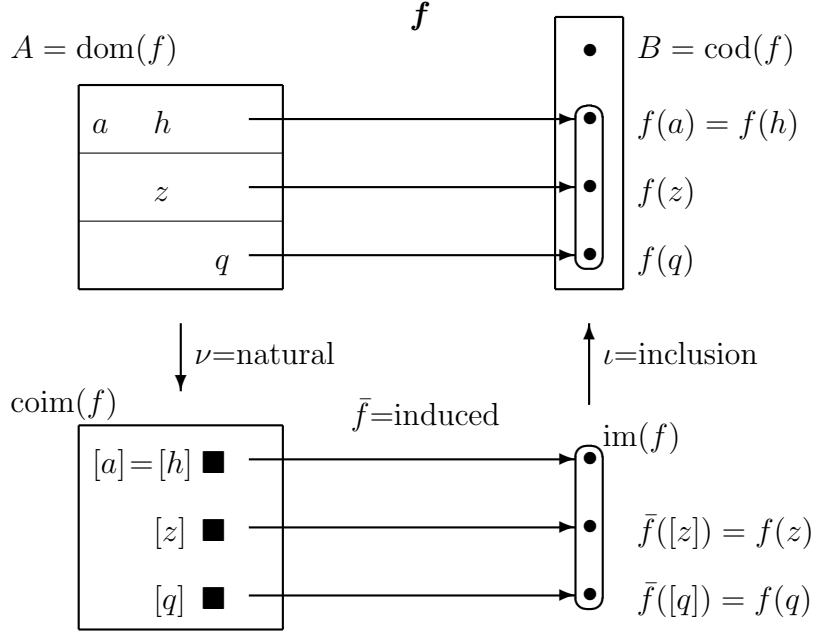


Terminology for functions.

Let A and B be sets and let $f: A \rightarrow B$ be a function from A to B . There are sets and functions related to A, B and f that have special names.



- (1) The *image* of f is $\text{im}(f) = f[A] = \{b \in B : \exists a \in A (f(a) = b)\}$. The image of a subset $U \subseteq A$ is $f[U] = \{b \in B : \exists u \in U (f(u) = b)\}$.
- (2) The *preimage* or *inverse image* of a subset $V \subseteq B$ is $f^{-1}[V] = \{a \in A : f(a) \in V\}$.
- (3) The preimage of a singleton $\{b\}$ is written $f^{-1}(b)$ and sometimes called the *fiber* of f over b . The fiber containing the element a is sometimes written $[a]$.
- (4) The *coimage* of f is the set $\text{coim}(f) = \{f^{-1}(b) : b \in \text{im}(f)\}$ of all nonempty fibers.
- (5) The *natural map* is $\nu: A \rightarrow \text{coim}(f): a \mapsto [a]$.
- (6) The *inclusion map* is $\iota: \text{im}(f) \rightarrow B: b \mapsto b$.
- (7) The *induced map* is $\bar{f}: \text{coim}(f) \rightarrow \text{im}(f): [a] \mapsto f(a)$.

Some facts:

- (1) The natural map is *surjective*.
- (2) The inclusion map is *injective*.
- (3) The induced map is *bijective*.
- (4) $f = \iota \circ \bar{f} \circ \nu$. (This is the *canonical factorization* of f .)

Practice problems.

- (1) Draw a figure like the previous one illustrating $f: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto x^2$. Identify all the “named” sets and functions.

- (2) Repeat the previous exercise for the function $g: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \mapsto x + y$.

- (3) Repeat for the *identity function* $\text{id}: A \rightarrow A: a \rightarrow a$.

- (4) Repeat for the *second coordinate projection* $\pi: X \times Y \rightarrow Y: (x, y) \rightarrow y$.

- (5) Show that
 - (a) the composition of two injective functions is injective,
 - (b) the composition of two surjective functions is surjective, and
 - (c) the composition of two bijective functions is bijective.

- (6) Show that injective functions are *left cancellable*: if f is injective, then $f \circ g = f \circ h$ implies $g = h$.

- (7) Show that surjective functions are *right cancellable*: if f is surjective, then $g \circ f = h \circ f$ implies $g = h$.

- (8) Show that if $f: A \rightarrow B$ is a function, then $f^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$ is also a function. Show that f is injective iff f^{-1} is surjective, and f is surjective iff f^{-1} is injective.