Disjunctive Normal Form!

(1) disjunction = $\vee$
(2) conjunction = $\wedge$
(3) Disjunctive Normal Form = $\bigvee(\bigwedge(\pm\text{variables}))$.
(4) Conjunctive Normal Form = $\bigwedge(\bigvee(\pm\text{variables}))$.

We have discussed truth tables for the connectives $\wedge, \vee, \neg, \to$, and $\leftrightarrow$, as well as for truth operations that can be constructed from these. Are there any truth operations we have missed? That is, is every truth operation constructible from $\wedge, \vee, \neg, \to$, and $\leftrightarrow$?

All truth operations of positive arity are constructible from any one of these sets of functions:

(1) $\{\wedge, \vee, \neg\}$, or
(2) $\{\vee, \neg\}$, or
(3) $\{\wedge, \neg\}$, or
(4) $\{\text{nand}\}$, where "x nand y" means $\neg(x \wedge y)$, or
(5) $\{\text{nor}\}$, where "x nor y" means $\neg(x \vee y)$, or
(6) $\{\to, \neg\}$.

These sets are examples of *complete* sets of truth operations.

Let's see why the connectives $\{\wedge, \vee, \neg\}$ are sufficient to construct any truth operation.

Claim 1. Any truth operation of the form $\bigwedge(\pm\text{variables})$ has only one 1 in its truth table. Conversely, any truth table of positive arity with only one 1 can be realized as the truth table of a truth operation of the form $\bigwedge(\pm\text{variables})$.

A *monomial* is an expression of the form "$\bigwedge(\pm\text{variables})$", that is, it is a conjunction of variables and negations of variables, with all variables appearing.

Claim 2. Any truth table of arity $n > 0$ can be realized as the truth table of a disjunction of monomials, that is, of a truth operation of the form $\bigvee(\bigwedge(\pm\text{variables}))$.

A *disjunctive normal form* of a truth operation in the variables $p_1, p_2, \ldots, p_k$ is any expression for the operation which has the form $\bigvee\left(\bigwedge_{i=1}^{k}(\pm p_i)\right)$.