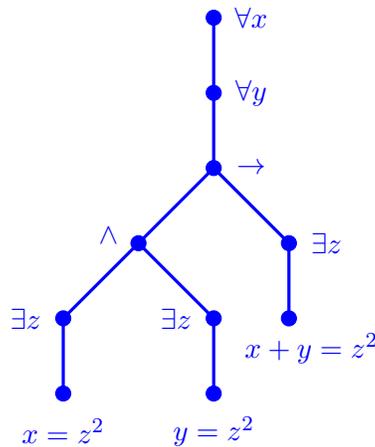


Solutions to HW 7.

1. This problem concerns the formal sentence

$$(\forall x)(\forall y)((\exists z)(x = z^2) \wedge ((\exists z)(y = z^2))) \rightarrow ((\exists z)(x + y = z^2)).$$

(i) Draw the formula tree for this sentence.



(ii) Standardize the variables apart. (There is more than one way to do this.)

$$(\forall x) (\forall y) (((\exists u) (x = u^2)) \wedge ((\exists v) (y = v^2))) \rightarrow ((\exists w) (x + y = w^2)).$$

(iii) Write the sentence in prenex form. (There is more than one way to do this.)

$$(\forall x) (\forall y) (\forall u) (\forall v) (\exists w) (((x = u^2) \wedge (y = v^2)) \rightarrow (x + y = w^2)).$$

2. This problem also concerns the formal sentence from Problem 1.

(i) Is the sentence true in the natural numbers, \mathbb{N} ? **No!** Give a winning strategy for the appropriate quantifier. (Appropriate quantifier is \forall .)

- \forall chooses $x = 1$.
- \forall chooses $y = 1$.
- \forall chooses $u = 1$.
- \forall chooses $v = 1$.
- To win, \exists would have to choose w so that $w^2 = 2$. There is no such $w \in \mathbb{N}$, so \exists loses.

(ii) Is the sentence true in the real numbers, \mathbb{R} ? **Yes!** Give a winning strategy for the appropriate quantifier. (Appropriate quantifier is \exists .)

- \forall chooses any x .
- \forall chooses any y .
- \forall chooses any u .
- \forall chooses any v .

\forall has already lost, unless the choices made satisfy $x = u^2$ and $y = v^2$, so assume that these equalities hold.

- To win, \exists would have to choose w so that $w^2 = x + y = u^2 + v^2$. So \exists can win by choosing $w = \sqrt{u^2 + v^2}$. This choice is possible, since in \mathbb{R} squares are nonnegative, so $u^2, v^2 \geq 0$. Also, in \mathbb{R} , a sum of nonnegative numbers is nonnegative, from which we get $u^2 + v^2 \geq 0$. Finally, any nonnegative real number has a real square root, so it is possible to choose a real number w satisfying $w = \sqrt{u^2 + v^2}$.

(iii) Negate the sentence, and then rewrite the negation so that it is in prenex form.

$$\neg(\forall x) (\forall y) (\forall u) (\forall v) (\exists w) (((x = u^2) \wedge (y = v^2)) \rightarrow (x + y = w^2)).$$

$$(\exists x) (\exists y) (\exists u) (\exists v) (\forall w) \neg(((x = u^2) \wedge (y = v^2)) \rightarrow (x + y = w^2)).$$

3. Show that the following pairs of propositions are logically equivalent. What does each equivalence say about proof strategies?

(i) $H \rightarrow C$ versus $(H \wedge (\neg C)) \rightarrow \perp$.

H	C	$H \rightarrow C$	$\neg C$	$H \wedge (\neg C)$	\perp	$(H \wedge (\neg C)) \rightarrow \perp$
0	0	1	1	0	0	1
0	1	1	0	0	0	1
1	0	0	1	1	0	0
1	1	1	0	0	0	1

This shows the equivalence between direct proof ($H \rightarrow C$) and proof by contradiction ($(H \wedge (\neg C)) \rightarrow \perp$).

(ii) $H \rightarrow C$ versus $((\neg C) \rightarrow (\neg H))$.

H	C	$H \rightarrow C$	$\neg C$	$\neg H$	$(\neg C) \rightarrow (\neg H)$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

This shows the equivalence between direct proof ($H \rightarrow C$) and proof of the contrapositive $((\neg C) \rightarrow (\neg H))$.

(iii) $(H_1 \wedge H_2) \rightarrow C$ versus $(\neg C) \rightarrow ((\neg H_1) \vee (\neg H_2))$.

H_1	H_2	C	$H_1 \wedge H_2$	$(H_1 \wedge H_2) \rightarrow C$	$\neg C$	$\neg H_1$	$\neg H_2$	$(\neg H_1) \vee (\neg H_2)$	$(\neg C) \rightarrow ((\neg H_1) \vee (\neg H_2))$
0	0	0	0	1	1	1	1	1	1
0	0	1	0	1	0	1	1	1	1
0	1	0	0	1	1	1	0	1	1
0	1	1	0	1	0	1	0	1	1
1	0	0	0	1	1	0	1	1	1
1	0	1	0	1	0	0	1	1	1
1	1	0	1	0	1	0	0	0	0
1	1	1	1	1	0	0	0	0	1

This shows the equivalence between direct proof with two hypotheses $((H_1 \wedge H_2) \rightarrow C)$ and proof of the contrapositive with two hypotheses $((\neg C) \rightarrow ((\neg H_1) \vee (\neg H_2)))$.