Solutions to HW 5.

In the first two HW problems we will use these recursive definitions. (Recall: $S(n) = n \cup \{n\}$ is the successor of n.)

Addition

$$\begin{array}{rll} m+0 & := m & \text{(Initial Condition for } +) \\ m+S(n) & := S(m+n) & \text{(Recurrence Relation (or Propagation Rule) for } +) \end{array}$$

Multiplication

$$\begin{array}{rll} m\cdot 0 & := 0 & \text{(Initial Condition for } \cdot) \\ m\cdot S(n) & := m\cdot n+m & \text{(Recurrence Relation (or Propagation Rule) for } \cdot) \end{array}$$

1. Prove that $m(nk) = (mn)k$ holds for the natural numbers.

This is a proof by induction on k. (We will use the result of Problem 2 here.)
(Base Case: $k = 0$)

$$\begin{array}{rll} m(n0) & = m0 & (\text{IC}, \cdot) \\ & = 0 & (\text{IC}, \cdot) \\ & = (mn)0 & (\text{IC}, \cdot) \end{array}$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{array}{rll} m(n\cdot S(k)) & = m(nk+n) & (\text{RR}, \cdot) \\ & = m(nk)+mn & \text{(Distributive Law)} \\ & = (mn)k+mn & \text{(IH)} \\ & = (mn)\cdot S(k) & (\text{RR}, \cdot) \end{array}$$

2. Prove that $m(n + k) = (mn) + (mk)$ holds for the natural numbers.

This is a proof by induction on k.
(Base Case: $k = 0$)

$$\begin{array}{rll} m(n+0) & = mn & (\text{IC}, +) \\ & = mn+0 & (\text{IC}, +) \\ & = mn+m0 & (\text{IC}, \cdot) \end{array}$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{array}{rll} m(n+S(k)) & = mS(n+k) & (\text{RR}, +) \\ & = m(n+k)+m & (\text{RR}, \cdot) \\ & = (mn+mk)+m & \text{(IH)} \\ & = mn+(mk+m) & (\text{Associative Law}, +) \\ & = mn+mS(k) & (\text{RR}, \cdot) \end{array}$$

3. Prove that if $k \in \mathbb{N}$, then any subset of k is finite.

Since the definition of "finite set" is "a set that is equipotent with a natural number", what we must establish is that:

Given $k \in \mathbb{N}$ and $X \subseteq k$, there exists $r \in \mathbb{N}$ and a bijection $f : r \to X$.

We will prove this by induction on k.

(Base Case: $k = 0$) If $k = 0 = \emptyset$ and $X \subseteq k$, then $X = \emptyset$. In this case, the empty function $f = \emptyset$ is a bijection from $r = 0 \in \mathbb{N}$ to X.

(Inductive Step: Assume true for k, prove true for $S(k)$)

Choose any subset $Y \subseteq S(k)$. We must find some $t \in \mathbb{N}$ and a bijection $g : t \to Y$.

Case 1. $Y \subseteq k$.

By the inductive hypothesis, there is an $r \in \mathbb{N}$ and a bijection $f : r \to Y$, so we can take $t = r$ and $g = f$.

Case 2. $Y \not\subseteq k$.

Since $Y \subseteq S(k) = k \cup \{k\}$, if $Y \not\subseteq k$, then $k \in Y$ and $Y - \{k\} \subseteq k$.

Use Case 1 to find $r \in \mathbb{N}$ and a bijection $f : r \to Y - \{k\}$. Now let $t = r \cup \{r\} = S(r) \in \mathbb{N}$ and define a bijection $g : S(r) \to Y$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in r \\ k & \text{if } x = r \end{cases}$$