Solutions to HW 4.

1. Show that the kernel of a function with domain A is an equivalence relation on A.

We must show that $\ker(f)$ is reflexive, symmetric, and transitive.

(1) For any $a \in A$, $f(a) = f(a)$, so $(a, a) \in \ker(f)$.

(2) For any $a, b \in A$, if $(a, b) \in \ker(f)$, then $f(a) = f(b)$. But then $f(b) = f(a)$, so $(b, a) \in \ker(f)$.

(3) For any $a, b, c \in A$, if $(a, b), (b, c) \in \ker(f)$, then $f(a) = f(b)$ and $f(b) = f(c)$. But then $f(a) = f(c)$, so $(a, c) \in \ker(f)$.

2. Show that if E is an equivalence relation on A, then E is the kernel of some function with domain A. (Hint: You need to find a function with domain A and kernel E. Let $P_E = \{[a] \in E \mid a \in A\}$ be the partition associated to E. Show that the natural map $\nu : A \to P_E : a \mapsto [a]$ has kernel E.)

Following the hint, let $\nu : A \to P_E$ be the function $a \mapsto [a]$. Then

$$\begin{aligned} (a, b) \in E &\iff [a] = [b] \\ &\iff \nu(a) = \nu(b) \\ &\iff (a, b) \in \ker(\nu). \end{aligned}$$

Since the sets E and $\ker(\nu)$ contain the same elements, they are equal.

3. Suppose that $f : A \to B$ and $g : A \to C$ are two functions with common domain A. Let $f \times g : A \to B \times C$ be the product function: $a \mapsto (f(a), g(a))$. Show that $\ker(f \times g) = \ker(f) \cap \ker(g)$.

$$\begin{aligned} (a, b) \in \ker(f \times g) &\iff (f \times g)(a) = (f \times g)(b) \\ &\iff (f(a), g(a)) = (f(b), g(b)) \\ &\iff f(a) = f(b) \text{ and } g(a) = g(b) \\ &\iff (a, b) \in \ker(f) \cap \ker(g). \end{aligned}$$