Solutions to HW 2.

1. Show that if A, B and C are sets, then $\{A, B, C\}$ is a set in each of the following ways:

(i) Using the Axiom of Replacement.

First construct a 3-element set, like $3 = \{0, 1, 2\}$. This can be constructed from the empty set by repeated use of the successor function. ($1 = S(0) = \{0\}, 2 = S(1) = \{0, 1\}, 3 = S(2) = \{0, 1, 2\}$.) Next use a function F which replaces the i-th number with the i-letter of the Latin alphabet: $0 \mapsto A, 1 \mapsto B, 2 \mapsto C$. Then, using the set $\{0, 1, 2\}$ and this replacement function we obtain that $\{F(0), F(1), F(2)\} = \{A, B, C\}$ is a set.

(ii) Without using the Axiom of Replacement.

By Pairing set A with set B we obtain that $\{A, B\}$ is a set. By Pairing set C with itself we obtain that $\{C\}$ is a set. Using Union we obtain that $\{A, B\} \cup \{C\} = \{A, B, C\}$ is a set.

2. Your friend offers a wager that, under the Kuratowski encoding, the ordered pair $(0, 1)$ equals the natural number three. Should you take the wager? Explain.

Take the wager!
$0 = \{\}, 1 = \{0\}, 2 = \{0, 1\}$ and $3 = \{0, 1, 2\}$. Thus $(0, 1) = \{\{0\}, \{0, 1\}\} = \{1, 2\} \neq \{0, 1, 2\} = 3$.

(Here is a second solution: any ordered pair (x, y), considered as a set $\{\{x\}, \{x, y\}\}$, contains one or two distinct elements. But 3 contains three distinct elements, so $(x, y) \neq 3$ for any x and y.)

3. Show that $\emptyset \times A = \emptyset$.

We must show that $\emptyset \times A$ has no elements. But this is clear, since if $(x, y) \in \emptyset \times A$, then $x \in \emptyset$, which is impossible ($\emptyset$ has no elements).