Binomial and Multinomial Coefficients!

(1) **Binomial coefficients**

(a) (Combinatorial interpretation) $\binom{n}{k} = C(n,k) =$ the number of ways to choose a k-element subset of an n-element set.

(b) (Formula) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

(c) (Recursion)

- $\binom{n}{0} = \binom{n}{n} = 1$.
- $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
- Note: $\binom{n}{k} = 0$ if $k < 0$ or $k > n$.

(d) (Theorem) $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}y^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k$.

(2) **Trinomial coefficients**

(a) (Combinatorial interpretation) $\binom{n}{k_1,k_2,k_3}$, when $n = k_1 + k_2 + k_3$, is the number of ways to choose a k_1-element subset of an n-element set, then a k_2-elements subset from the $n - k_1$ remaining elements, then a k_3-element subset from the remaining $n - k_1 - k_2$ elements. Or, it is the number of ways to distribute n distinct balls to 3 distinct boxes with the ith box receiving k_i balls.

(b) (Formula) $\binom{n}{k_1,k_2,k_3} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3} = \frac{n!}{k_1!k_2!k_3!}$.

(c) (Recursion)

- $\binom{n}{n,0,0} = \binom{n}{0,n,0} = \binom{n}{0,0,n} = 1$.
- $\binom{n}{k_1,k_2,k_3} = \binom{n-1}{k_1-1,k_2,k_3} + \binom{n-1}{k_1,k_2-1,k_3} + \binom{n-1}{k_1,k_2,k_3-1}$.
- Note: $\binom{n}{k_1,k_2,k_3} = 0$ if any $k_i < 0$.

(d) (Theorem) $(x+y+z)^n = \sum_{k_1+k_2+k_3=n} \binom{n}{k_1,k_2,k_3} x^{k_1}y^{k_2}z^{k_3}$.

(3) **Multinomial coefficients**

(a) (Combinatorial interpretation) $\binom{n}{k_1,k_2,\ldots,k_r}$, when $n = k_1 + k_2 + \cdots + k_r$, is the number of ways to distribute n distinct balls to r distinct boxes, with the ith box receiving k_i balls.

(b) (Formula) $\binom{n}{k_1,k_2,\ldots,k_r} = \frac{n!}{k_1!k_2!\cdots k_r!}$.

(c) (Recursion)

- $\binom{n}{0,\cdots,n,\cdots,0} = 1$.
- $\binom{n}{k_1,k_2,\ldots,k_r} = \binom{n-1}{k_1-1,k_2,\ldots,k_r} + \binom{n-1}{k_1,k_2-1,\ldots,k_r} + \cdots + \binom{n-1}{k_1,k_2,\ldots,k_r-1}$.
- Note: $\binom{n}{k_1,k_2,\ldots,k_r} = 0$ if any $k_i < 0$.

(d) (Theorem) $(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+x_r=n} \binom{n}{k_1,k_2,\ldots,k_r} x_1^{k_1}x_2^{k_2}\cdots x_r^{k_r}$.

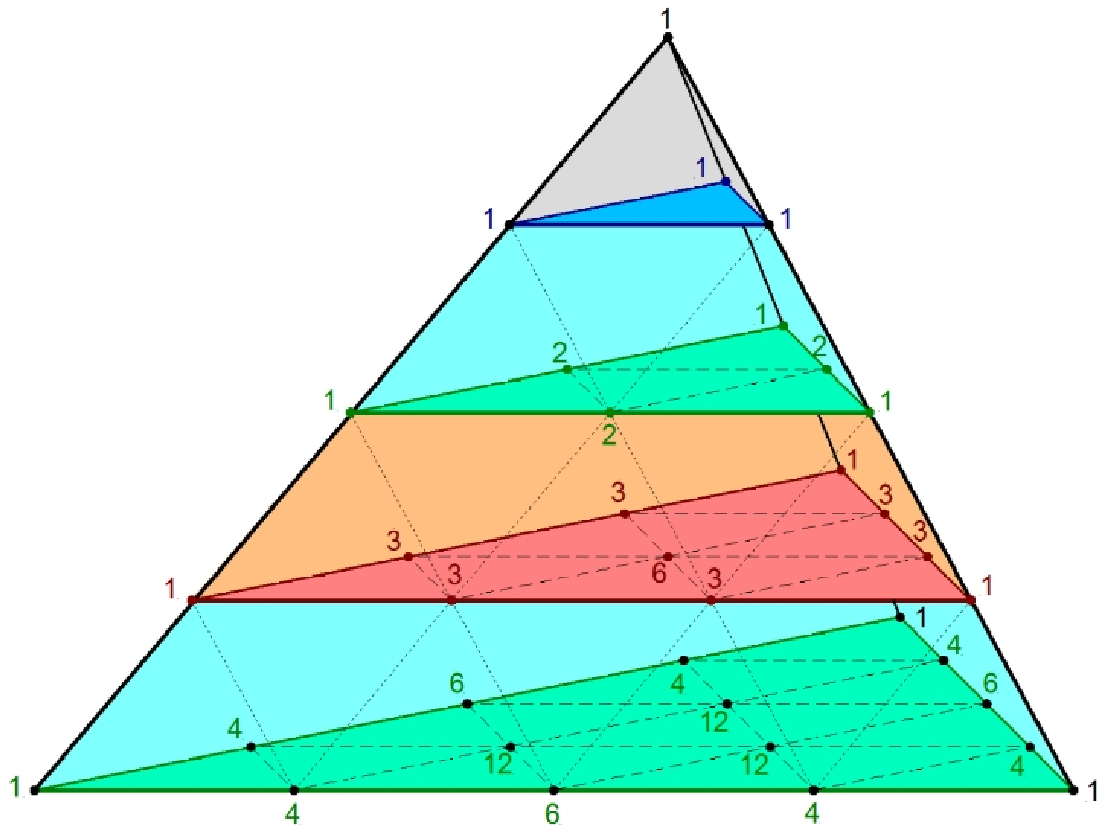

FIGURE 1. Pascal's Pyramid: $\binom{n}{k_1,k_2,k_3}$