The Hilbert Field.

Thm.

(1) $\mathbb{R}$ is a Pythagorean ordered field (POF).
(2) The intersection of any collection of Pythagorean ordered subfields of $\mathbb{R}$ is also a POF.

(The intersection of all Pythagorean ordered subfields of $\mathbb{R}$ is called the **Hilbert field**, Ω.)

Sketch of Proof of Item (2). Assume that $\mathcal{P} = \{\mathbb{R}, \mathbb{S}, \mathbb{T}, \ldots\}$ is a collection of POF-substructures of $\mathbb{R}$, and that $\mathbb{I}$ is the intersection. We have to show that $\mathbb{I}$ is a POF-substructure. This requires showing that $\mathbb{I}$ contains $0, 1$, and $\mathbb{I}$ is closed under the field operations, and that $\mathbb{I}$ is closed under the Pythagorean operation: $p(x, y) = +\sqrt{x^2 + y^2}$. All arguments are based on the same idea, so I will just explain why (i) $0 \in \mathbb{I}$ and why (ii) $\mathbb{I}$ is closed under $+$.

For (i), note that every element of the collection $\mathcal{P}$ is a POF-substructure of $\mathbb{R}$, so every one contains 0, so the intersection, $\mathbb{I}$, also contains 0.

For (ii), we choose $a, b \in \mathbb{I}$ and argue that $a + b \in \mathbb{I}$. Since $a, b \in \mathbb{I}$, both a and b belong to every POF in $\mathcal{P}$. Hence $a + b \in \mathbb{R}$ also belongs to every POF in $\mathcal{P}$. Hence $a + b$ belongs to the intersection $\mathbb{I}$ of the structures in $\mathcal{P}$. $\square$.

It is easy to construct some elements of the Hilbert field Ω:

(1) Ω is closed under $+, -, 0, 1$, so $\mathbb{Z} \subseteq \Omega$.
(2) Ω is closed under multiplication and inversion of nonzero elements, so $\mathbb{Q} \subseteq \Omega$.
(3) By using the Pythagorean operation you can show that some irrational numbers belong to Ω: $p(1, 1) = \sqrt{1^2 + 1^2} = \sqrt{2} \in \Omega$. $p(1, \sqrt{2}) = \sqrt{3} \in \Omega$.

Question. Are the following in Ω: $\sqrt{1 + \sqrt{2}}$? $\sqrt{7 + 2\sqrt{5 + \sqrt{6}}}$? $\sqrt[3]{2}$? π?

The answer to this question is complicated by the fact that some numbers are expressible in multiple ways. For example, $\sqrt{2}+\sqrt{3} = \sqrt{5 + \sqrt{6}}$, and $\sqrt[3]{2 + \sqrt{-121}}+\sqrt[3]{2 - \sqrt{-121}} = 4$. This means that it is hard to tell if a number belongs to Ω just by looking at it.

Algebraic conjugates.

An **automorphism** of a field $\mathbb{F} = \langle F; +, -, 0, \cdot 1\rangle$ is an invertible homomorphism from the field to itself, $\alpha\colon \mathbb{F} \to \mathbb{F}$. This means that the following hold:

(1) α is a 1-1 and onto function from $\mathbb{F}$ to itself.
(2) $\alpha(x+y) = \alpha(x) + \alpha(y)$, $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$, $\alpha(-x) = -\alpha(x)$, $\alpha(0) = 0$, $\alpha(1) = 1$.

For example, complex conjugation $\alpha(a + bi) = a - bi$ is a function $\alpha\colon \mathbb{C} \to \mathbb{C}$ that is an automorphism of the field of complex numbers.

An element of the complex numbers is **algebraic** if it is a root of a nonzero rational polynomial. Otherwise it is **transcendental**. If z is an algebraic complex number, then its **minimal polynomial** is the least degree monic rational polynomial that it satisfies. Two complex numbers $z, w \in \mathbb{C}$ are **algebraic conjugates** if there is an automorphism $\alpha\colon \mathbb{C} \to \mathbb{C}$ such that $\alpha(z) = w$. It is a theorem that two algebraic numbers are algebraic conjugates iff their minimal polynomials are equal. A complex number is **totally real** if all of its algebraic conjugates are real.

Thm. Any element of Ω

(1) is algebraic,
(2) has minimal polynomial whose degree is a power of two, and
(3) is totally real.

Now let's return to:

Question. Are the following in Ω: $\sqrt{1+\sqrt{2}}$? $\sqrt{7+2\sqrt{5+\sqrt{6}}}$? $\sqrt[3]{2}$? π?