

Communication

On the restricted homomorphism problem

Richard C. Brewster*, Timothy Graves

Department of Mathematics and Statistics, Thompson Rivers University, PO Box 3010, Kamloops, BC, Canada, V2C 5N3

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Abstract

The *restricted homomorphism problem* $RHP(H, Y)$ asks: given an input digraph G and a homomorphism $g : G \rightarrow Y$, does there exist a homomorphism $f : G \rightarrow H$? We prove that if H is *hereditarily hard* and $Y \rightarrow H$, then $RHP(H, Y)$ is NP-complete. Since non-bipartite graphs are hereditarily hard, this settles in the affirmative a conjecture of Hell and Nešetřil.

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1. Introduction

We consider finite graphs and digraphs. Loops are allowed, but not parallel edges. To unify the approach, definitions and results are stated in terms of digraphs, as is done in [11]. Graphs are viewed as digraphs where each edge has been replaced by two oppositely directed arcs, i.e. graphs are viewed as symmetric digraphs.

Given digraphs G and H , a *homomorphism of G to H* is a mapping $f : V(G) \rightarrow V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. We write $G \rightarrow H$ to indicate the existence of a homomorphism of G to H , and $f : G \rightarrow H$ when we wish to explicitly name the homomorphism. Since the homomorphic image of a symmetric digraph is a symmetric digraph, the definition applies to graphs. Formally we define the following decision problem for a fixed digraph H .

HOMOMORPHISM $HOM(H)$.

INSTANCE: A digraph G .

QUESTION: Does G admit a homomorphism to H ?

Many familiar problems from graph theory can be expressed in the language of homomorphisms; however, they are, as is $HOM(H)$, notoriously difficult in terms of their computational complexity.

A celebrated result of Hell and Nešetřil [10] states that for *graphs*, $HOM(H)$ is polynomial time solvable if H is bipartite or contains a loop, and is NP-complete otherwise. For digraphs, such a dichotomy theorem is not

* Corresponding author. Tel.: +1 250 828 5215; fax: +1 250 371 5675.

E-mail address: rbrewster@tru.ca (R.C. Brewster).

known, nor does there exist a plausible conjecture about which problems are NP-complete and which are polynomial. Feder and Vardi [6] have shown that $HOM(H)$ for digraphs is as rich in terms of computational complexity as all of CSP (constraint satisfaction problems). Nonetheless, large families of digraphs have been identified for which the complexity of $HOM(H)$ is known. One such family is the *hereditarily hard* digraphs. A digraph H is hereditarily hard if for any loop-free digraph H' containing H as a subgraph, $HOM(H')$ is NP-complete. This family was introduced in [1], and recent work has been done in [8,15]. The result of Hell and Nešetřil can be restated to say that non-bipartite graphs, and in particular odd cycles, are hereditarily hard.

Given the difficult nature of homomorphism problems, a natural question to ask is whether restricting the inputs to $HOM(H)$ makes the problem easier. Füredi, Griggs, and Kleitman [7] posed the question “What is the complexity of $HOM(C_5)$ if the instances are restricted to 3-colourable graphs?”. Note that any graph that maps to C_5 must map to K_3 by composition. Hence, this is a natural restriction. It turns out that $HOM(C_5)$ is NP-complete even when a 3-colouring of the input is provided as part of the instance [2]. The general situation can be captured by the following formal decision problem which is the focus of [4], cf. [3]. Let H and Y be fixed digraphs.

RESTRICTED HOMOMORPHISM $RHP(H, Y)$.

INSTANCE: A digraph G and a homomorphism $g : G \rightarrow Y$.

QUESTION: Does G admit a homomorphism to H ?

The complexity of $RHP(H, Y)$ was conjectured as follows:

Conjecture 1 (Hell and Nešetřil, 1992). *Let H and Y be graphs. Then $RHP(H, Y)$ is polynomial in the following cases:*

- H contains a loop,
- H is bipartite, or
- $Y \rightarrow H$;

otherwise, $RHP(H, Y)$ is NP-complete.

We shall prove [Conjecture 1](#).

It is easy to see that for graphs $RHP(H, Y)$ is polynomial time solvable if H contains a loop or is bipartite since $HOM(H)$ itself is polynomial time solvable, or if $Y \rightarrow H$ in which case the problem is trivial. The following example from [4] demonstrates that the conjecture does not naturally extend to digraphs. Let R_4 be the digraph obtained from the transitive tournament on four vertices by reversing the arc from the source to the sink. It turns out, see [4], that $RHP(R_4 \times C_2, R_4)$ is polynomial time solvable although $HOM(R_4 \times C_2)$ is NP-complete and $R_4 \not\rightarrow R_4 \times C_2$.

2. Main results

Let H and Y be digraphs. The categorical product $H \times Y$ is the digraph with vertex set $V(H) \times V(Y)$ and edge set defined by $(h_1, y_1)(h_2, y_2) \in E(H \times Y)$ if $h_1h_2 \in E(H)$ and $y_1y_2 \in E(Y)$. The exponential digraph H^Y has as its vertex set all functions $V(Y) \rightarrow V(H)$, with $fg \in E(H^Y)$ if for all $uv \in E(Y)$ we have $f(u)g(v) \in E(H)$; see [11]. Some fundamental properties of products and the exponential digraphs are stated below.

Proposition 2 ([11]). *Let H and Y be digraphs. Then*

1. *for all digraphs K , $K \rightarrow H \times Y$ if and only if $K \rightarrow H$ and $K \rightarrow Y$;*
2. *H is an induced subgraph of H^Y ;*
3. *$Y \rightarrow H$ if and only if H^Y contains a loop; and*
4. *for all digraphs K , $K \rightarrow H^Y$ if and only if $K \times Y \rightarrow H$.*

Since a non-bipartite graph without a loop is hereditarily hard, [Conjecture 1](#) follows from the theorem below.

Theorem 3. *Let H be a hereditarily hard digraph. Suppose Y is a digraph such that $Y \not\rightarrow H$. Then $RHP(H, Y)$ is NP-complete.*

Proof. Clearly $RHP(H, Y)$ is in NP. Consider the exponential digraph H^Y . Since $Y \rightarrow H$, the digraph H^Y is loop-free. Also, H is an induced subgraph of H^Y ; hence using hereditary hardness, $HOM(H^Y)$ is NP-hard.

Let K be an instance of $HOM(H^Y)$. We consider the digraph $K \times Y$ and the projection $\pi_2 : K \times Y \rightarrow Y$ as an instance of $RHP(H, Y)$. By Proposition 2, $K \rightarrow H^Y$ if and only if $K \times Y \rightarrow H$. Thus, K is a YES instance of $HOM(H^Y)$ if and only if $(K \times Y, \pi_2)$ is a YES instance of $RHP(H, Y)$. \square

The exponential digraph was introduced in the study of multiplicative graphs by El-Zahar and Sauer [5]. (More about its history can be found in [11].) It was also studied in a series of papers on duality in the category Graph by Nešetřil and Tardif [12,13]. The resulting connection between products and exponentiation developed over the 1990s, since the formulation of Conjecture 1, and as surveyed in [11] (cf. Proposition 2) provides us with the tools for the short proof of our main result.

3. Applications

We first apply Theorem 3 to circular and fractional colourings (as defined, say, in [11]). For circular colourings, we easily derive the following corollary.

Corollary 4. *Let p and q be positive integers such that $p/q > 2$. Then for any $\epsilon > 0$, it is NP-complete to determine whether an input graph G admits a (p', q') -colouring for $2 < p'/q' < p/q - \epsilon$, even when a (p, q) -colouring of G is provided as part of the input.*

Indeed, a (p, q) -colouring of G is a homomorphism of G to a certain graph $K_{p/q}$ [11], which is loop-free and non-bipartite (as long as $p/q > 2$). In fact, since this graph is a circulant graph, and Conjecture 1 has been verified for circulant graphs in [4], the corollary follows from [4]. It has also been independently shown in [9], see also [14].

In particular, determining whether $\chi_c(G) \leq n - \frac{1}{k}$ (for sufficiently large k) is NP-complete even when an n -colouring of G is provided as part of the instance.

By the same token Theorem 3 implies an analogous result about fractional colourings.

Corollary 5. *Let p and q be positive integers such that $p/q > 2$. Then for any $\epsilon > 0$, it is NP-complete to determine whether an input graph G admits a p'/q' -fractional colouring for $2 < p'/q' < p/q - \epsilon$, even when a p/q -fractional colouring of G is provided as part of the input.*

Again, a p/q -fractional colouring of G is a homomorphism of G to the Kneser graph $K(p, q)$ [11], which is loop-free and non-bipartite (as long as $p/q > 2$).

We again observe that in particular determining whether $\chi_f(G) \leq n - \frac{1}{k}$ (for sufficiently large k) is NP-complete even when an n -colouring of G is provided as part of the instance.

Finally we comment on hereditary hardness. Recall the example in [4] demonstrating Conjecture 1 does not naturally extend to digraphs. The assumption $NP \neq P$ implies $R_4 \times C_2$ is not hereditarily hard. It would be interesting to pursue this method in an attempt to identify large families of non-hereditarily hard digraphs.

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