

Theorem 1. (*Ultrafilter Convergence Theorem*) Let $\mathcal{X} = \langle X; \mathcal{T} \rangle$ be a topological space.

- (1) If $Y \subseteq X$ and $\ell \in X$, then $\ell \in \overline{Y}$ iff there is an ultrafilter \mathcal{U} on some nonempty set I and a function $f: I \rightarrow X$ such that
 - (a) $f(I) \subseteq Y$, and
 - (b) $\lim_{\mathcal{U}} f = \ell$.
- (2) \mathcal{X} is compact iff for every ultrafilter \mathcal{U} on a set I and every function $f: I \rightarrow X$ there is at least one $\ell \in X$ such that $\lim_{\mathcal{U}} f = \ell$.
- (3) \mathcal{X} is Hausdorff iff for every ultrafilter \mathcal{U} on a set I and every function $f: I \rightarrow X$ there is at most one $\ell \in X$ such that $\lim_{\mathcal{U}} f = \ell$.

Proof. [(1), \Rightarrow] Assume that $\ell \in \overline{Y}$. Let I be the set of open nhoods of ℓ in \mathcal{X} . Order I by reverse inclusion: $U \ll V$ iff $U \supseteq V$. The collection of principal order filters¹ in this up-directed poset has the FIP, so it generates a proper filter \mathcal{F} on I , which may be extended to an ultrafilter \mathcal{U} . For each $U \in I$ choose a point $y_U \in U \cap Y$; such a point exists since $\ell \in \overline{Y}$. Define $f: I \rightarrow X: U \mapsto y_U$.

Clearly $f(I) = \{y_U \mid U \in I\} \subseteq Y$, so (a) holds. To verify (b), choose an open nhood U of ℓ , and consider $f^{-1}(U)$. The set U contains y_V for every V satisfying $U \ll V$, so $f^{-1}(U)$ contains $[U]$, which is an element of \mathcal{F} and therefore also an element of \mathcal{U} . This proves (a).

[(1), \Leftarrow] We assume (a) and (b) and show that every open set O containing ℓ intersects Y . Since $\lim_{\mathcal{U}} f \stackrel{(b)}{=} \ell$ we have $f^{-1}(O) \in \mathcal{U}$. Since $\emptyset \notin \mathcal{U}$ we have $f^{-1}(O) \neq \emptyset$. Hence

$$\emptyset \neq f^{-1}(O) = f^{-1}(O) \cap I \stackrel{(a)}{=} f^{-1}(O) \cap f^{-1}(Y) = f^{-1}(O \cap Y),$$

which implies that $O \cap Y \neq \emptyset$.

[(2), \Rightarrow] Assume that \mathcal{X} is compact, but there is some ultrafilter \mathcal{U} on some set I and some function $f: I \rightarrow X$ with no ℓ satisfying $\lim_{\mathcal{U}} f = \ell$. Each $x \in X$ must have an open neighborhood O_x such that $f^{-1}(O_x) \notin \mathcal{U}$. By compactness, the open cover $\{O_x \mid x \in X\}$ has a finite subcover $\{O_1, \dots, O_n\}$. This yields $\cup_{i=1}^n f^{-1}(O_i) = f^{-1}(\cup_{i=1}^n O_i) = I \in \mathcal{U}$. We claim that if a finite union of subsets of I lies in \mathcal{U} then at least one summand lies in \mathcal{U} . Since no set $f^{-1}(O_i)$ lies in \mathcal{U} , this claim produces a contradiction.

We prove the contrapositive of the claim just mentioned, namely that $V, W \notin \mathcal{U}$ implies $V \cup W \notin \mathcal{U}$. If $V, W \notin \mathcal{U}$, then $X \setminus V, X \setminus W \in \mathcal{U}$, hence $(X \setminus V) \cap (X \setminus W) = X \setminus (V \cup W) \in \mathcal{U}$. Therefore $V \cup W \notin \mathcal{U}$.

[(2), \Leftarrow] Choose a collection \mathcal{C} of closed subsets of \mathcal{X} with the FIP. Let $I = X$ and let \mathcal{F} be the filter on I generated by \mathcal{C} . Let \mathcal{U} be an ultrafilter extending \mathcal{F} . Let $f: I \rightarrow X$ be the identity function. Choose ℓ so that $\lim_{\mathcal{U}} f = \ell$. Let \mathcal{N}_{ℓ} be the set of open neighborhoods of ℓ . If $O \in \mathcal{N}_{\ell}$, then $O = f^{-1}(O) \in \mathcal{U}$. This yields $\mathcal{C} \cup \mathcal{N}_{\ell} \subseteq \mathcal{U}$, showing that $\mathcal{C} \cup \mathcal{N}_{\ell}$ has the FIP. This proves that ℓ belongs to every closed set in \mathcal{C} , hence that $\ell \in \bigcap \mathcal{C}$.

[(3), \Rightarrow] Assume \mathcal{X} is Hausdorff and that $\lim_{\mathcal{U}} f = \ell$ and $\lim_{\mathcal{U}} f = m$ for appropriate \mathcal{U}, f, ℓ, m with $\ell \neq m$. Choose disjoint open neighborhoods O_{ℓ} and O_m of ℓ and m respectively. Then $f^{-1}(O_{\ell})$ and $f^{-1}(O_m)$ are disjoint sets in \mathcal{U} , a contradiction.

¹sets of the form $[U] := \{V \in \mathcal{T} \mid U \ll V\}$

[(3), \Leftarrow] Now suppose that \mathcal{X} is not Hausdorff and that $\ell \neq m$ are elements of \mathcal{X} that cannot be separated by disjoint open sets. If \mathcal{N}_ℓ and \mathcal{N}_m are the open neighborhood systems of these two points, then $\mathcal{N}_\ell \cup \mathcal{N}_m$ has the FIP. Let $I = X$, let \mathcal{U} be an ultrafilter on I containing $\mathcal{N}_\ell \cup \mathcal{N}_m$, and let $f: I \rightarrow X$ be the identity function. Both $\lim_{\mathcal{U}} f = \ell$ and $\lim_{\mathcal{U}} f = m$ hold, establishing the contrapositive of (2), \Leftarrow . \square