

Ultrafilters.

Definition 1. A **filter** on I is a nonempty set $\mathcal{F} \subseteq \mathcal{P}(I)$ such that

- (1) \mathcal{F} is closed under finite intersection, and
- (2) \mathcal{F} is closed under the formation of supersets (if $U \in \mathcal{F}$ and $U \subseteq V$, then $V \in \mathcal{F}$).

\mathcal{F} is **improper** if $\mathcal{F} = \mathcal{P}(I)$ and is **trivial** if $\mathcal{F} = \{I\}$.

It follows from Definition 1 (2) that \mathcal{F} is improper iff $\emptyset \in \mathcal{F}$.

Lemma 2. *The set of (proper) filters on I is closed under intersection and unions of directed families.*

Proof. The fact that filters are defined by closure properties guarantees that the set of (proper) filters on I is closed under intersection.

Let $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ be a Λ -directed family of filters on I . If $U_1, \dots, U_k \in \bigcup \mathcal{F}_\lambda$, then these sets belong to \mathcal{F}_μ for some μ . But then

$$U_1 \cap \dots \cap U_k \in \mathcal{F}_\mu \subseteq \bigcup \mathcal{F}_\lambda.$$

Now suppose that $U \in \bigcup \mathcal{F}_\lambda$ and $U \subseteq V$. Then $U \in \mathcal{F}_\mu$ for some μ and therefore $V \in \mathcal{F}_\mu \subseteq \bigcup \mathcal{F}_\lambda$. This shows that the union of a directed family of filters is a filter.

If $\bigcup \mathcal{F}_\lambda$ is improper, then it contains \emptyset . It must be that $\emptyset \in \mathcal{F}_\mu$ for some μ , and this \mathcal{F}_μ is also improper. \square

For any set $\mathcal{S} \subseteq \mathcal{P}(I)$ there is a least filter containing \mathcal{S} , namely the intersection of all filters containing \mathcal{S} . This filter is called the **filter generated by \mathcal{S}** and denoted $\langle \mathcal{S} \rangle$.

Lemma 3. *If $\mathcal{S} \subseteq \mathcal{P}(I)$, then TFAE.*

- (1) $U \in \langle \mathcal{S} \rangle$.
- (2) U contains a finite intersection of elements of \mathcal{S} .

Proof. [(1) \Rightarrow (2)] The set of all those sets which contain a finite intersection of elements of \mathcal{S} is a filter containing \mathcal{S} .

[(2) \Rightarrow (1)] Any filter containing \mathcal{S} must contain every set that contains a finite intersection of elements of \mathcal{S} by Definition 1 (1) and (2). \square

Definition 4. $\mathcal{S} \subseteq \mathcal{P}(I)$ has the **finite intersection property (FIP)** if any finite intersection of elements of \mathcal{S} is nonempty.

Hence \mathcal{S} has the finite intersection property iff $\langle \mathcal{S} \rangle$ is proper.

Definition 5. A proper filter \mathcal{F} on I is an **ultrafilter** if for every $U \subseteq I$ either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$.

Lemma 6. (1) *A filter \mathcal{F} on I is maximal under inclusion among proper filters on I iff it is an ultrafilter.*

- (2) *(Ultrafilter Lemma) Every proper filter on I can be extended to an ultrafilter.*

(3) *(Strengthening of (2)) Every proper filter on I is the intersection of the ultrafilters that extend it.*

Proof. [Proof of (1)] Assume \mathcal{F} is a maximal filter. If $U \notin \mathcal{F}$, then $\langle \mathcal{F} \cup \{U\} \rangle$ is improper, hence contains \emptyset . By Lemma 3, there is some $V \in \mathcal{F}$ such that $\emptyset = U \cap V$. Replacing V by a superset if necessary we may assume that $V = I \setminus U$. Hence $U \notin \mathcal{F}$ implies that $I \setminus U \in \mathcal{F}$, showing that maximal filters are ultrafilters.

For the other direction in (1), observe that any proper extension of an ultrafilter must contain some set and its complement, hence must contain \emptyset . Thus ultrafilters are maximal.

[Proof of (2)] If \mathcal{F} is a filter satisfying $\emptyset \notin \mathcal{F}$, then Zorn's Lemma guarantees that \mathcal{F} can be extended to a filter \mathcal{F}' that is maximal for $\emptyset \notin \mathcal{F}'$. (This uses the first part of Lemma 2.) By (1), any filter maximal for $\emptyset \notin \mathcal{F}'$ is an ultrafilter.

[Proof of (3)] Suppose that \mathcal{F} is proper and $U \notin \mathcal{F}$. By repeating the argument from the first paragraph of part (1) we see that $\mathcal{F}' := \langle \mathcal{F} \cup \{I \setminus U\} \rangle$ is a proper filter. Extend \mathcal{F}' to an ultrafilter \mathcal{U} using part (2). Since \mathcal{U} must contain $I \setminus U$ it cannot contain U . This shows that whenever $U \notin \mathcal{F}$ there is an ultrafilter \mathcal{U} extending \mathcal{F} satisfying $U \notin \mathcal{U}$. Hence the intersection of the ultrafilters containing \mathcal{F} is a filter containing \mathcal{F} which contains no sets not in \mathcal{F} , i.e. $\mathcal{F} = \bigcap_{\mathcal{U} \supseteq \mathcal{F}} \mathcal{U}$ \square