

Definition 1. Let I be an infinite set. An ultrafilter \mathcal{U} on I is *regular* if there is a subset $E \subseteq \mathcal{U}$ such that

- (i) $|E| = |I|$, and
- (ii) each $i \in I$ is contained in only finitely many elements of E .

Item (ii) is equivalent to the statement that any infinite subset of E has empty intersection. Of course, any finite subset of E has intersection in \mathcal{U} , so is nonempty.

Theorem 2. Any infinite set supports a regular ultrafilter.

Proof. It suffices to show that on any infinite set I there is a set $E \subseteq \mathcal{P}(I)$ satisfying (i) $|E| = |I|$, and (ii)' E has the FIP, but any infinite subset of E has empty intersection. For if (i) and (ii)' hold, then the Ultrafilter Lemma allows us to find a (necessarily regular) ultrafilter extending E .

Start with a set J whose size is the desired size of the index set. Let I be the set of finite subsets of J . Then $|I| = |J|$, as needed. For each $j \in J$ define $\hat{j} = \{i \in I \mid j \in i\}$, and then take $E = \{\hat{j} \mid j \in J\}$. Clearly (i) $|E| = |J| = |I|$ holds, since the function $j \mapsto \hat{j}$ is bijective.

To prove (ii)', realize that if $E_0 \subseteq E$, then $E_0 = \{\hat{j} \mid j \in J_0\}$ for some subset $J_0 \subseteq J$ satisfying $|J_0| = |E_0|$. Moreover, $\bigcap E_0 = \bigcap_{j \in J_0} \hat{j} = \{i \in I \mid J_0 \subseteq i\}$. If E_0 is finite then this intersection is nonempty, while if it is infinite the intersection is empty. \square

Theorem 3. (Frayne, Morel, Scott) If A is an infinite set and \mathcal{U} is a regular ultrafilter on I , then $|\prod_{\mathcal{U}} A| = |A|^{|I|}$.

Proof. Since $\prod_{\mathcal{U}} A$ is a quotient of A^I it follows that $|\prod_{\mathcal{U}} A| \leq |A|^{|I|}$.

For the reverse inequality, choose $E \subseteq \mathcal{U}$ witnessing regularity. Let B be the set of finite sequences of elements of A . Since A is infinite, we have $|B| = |A|$. To prove the desired inequality it will suffice to exhibit a 1-1 function $\alpha: A^E \rightarrow \prod_{\mathcal{U}} B = (B^I)/\theta_{\mathcal{U}}$. We will explain how to assign to any $f: E \rightarrow A$ a function $\hat{f}: I \rightarrow B$ such that $f \neq g$ implies $\hat{f} \not\equiv_{\mathcal{U}} \hat{g}$. Then we take α to be $f \mapsto [\hat{f}]_{\theta_{\mathcal{U}}}$.

Now we start explaining: Linearly order E with the relation $<$. Choose $f \in A^E$, and define $\hat{f} \in B^I$ as follows: if $i \in I$, let (e_1, \dots, e_m) be the finite set of all $e \in E$ such that $i \in e$ ordered according to the relation $<$. Define $\hat{f}(i) = (f(e_1), \dots, f(e_m)) \in B$.

It remains to check that $f \neq g$ implies $\hat{f} \not\equiv_{\mathcal{U}} \hat{g}$. Suppose that $f(e) \neq g(e)$ for some $e \in E$. For any $i \in e$, the element e will occur in the sequence (e_1, \dots, e_m) , say $e = e_k$. Now $\hat{f}(i) = (\dots, f(e_k), \dots) \neq (\dots, g(e_k), \dots) = \hat{g}(i)$. This shows that $\hat{f}(i) \neq \hat{g}(i)$ for all $i \in e$. Since $e \in \mathcal{U}$ this means $\hat{f} \not\equiv_{\mathcal{U}} \hat{g}$. \square