

Unreasonable exercise on structures.

Recall that we defined a plane geometry to be a 2-sorted structure of the form

$$\Pi = \langle \mathcal{P}, \mathcal{L}; \mathcal{I}, \mathcal{B}, \bar{\mathcal{C}}, \mathcal{C}_{\triangleleft} \rangle,$$

where

- (i) \mathcal{P} is a set (of “points”),
- (ii) \mathcal{L} is a set (of “lines”),
- (iii) $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ is a binary relation (incidence),
- (iv) $\mathcal{B} \subseteq \mathcal{P} \times \mathcal{P} \times \mathcal{P} = \mathcal{P}^3$ is a ternary relation (betweenness),
- (v) $\bar{\mathcal{C}} \subseteq \mathcal{P}^4$ is a 4-ary relation, and
- (vi) $\mathcal{C}_{\triangleleft} \subseteq \mathcal{P}^6$ is a 6-ary relation,

and (as part of the definition of “plane”) some axioms are satisfied.

For example, the axiom that any two distinct points determine a unique line might be written as

$$\forall P \forall Q ((P \neq Q) \rightarrow \exists \ell ((\mathcal{I}(P, \ell) \wedge \mathcal{I}(Q, \ell)) \wedge \forall m ((\mathcal{I}(P, m) \wedge \mathcal{I}(Q, m)) \rightarrow (m = \ell))))$$

Although we are not very far into the subject yet, this exercise asks that you show that certain common mathematical objects can be defined so that they are “structures” satisfying some axioms.

- (1) Represent each one of the following types of objects as “structures” of some kind: groups, vector spaces, G -sets, graphs, posets, topological spaces.

- (2) Write some formal sentences down which make sense for your structures, e.g., some of the axioms that define the objects they were invented to model. (Suggestions: write a sentence that makes sense for your “group” type structure that says multiplication is associative. Write a sentence down which says for posets that between any two comparable elements lies a third element. Write the axioms for topological spaces.)