Our goal is to prove Łos's Theorem, which asserts that a formula is satisfied by a tuple in an ultraproduct iff it is satisfied in almost every coordinate modulo $\mathcal{U}$. In order to compare satisfaction in the ultraproduct with satisfaction in a coordinate structure we refer the following diagram:

$$\begin{array}{ccccc} X = \{x_1, x_2, \ldots\} & \xrightarrow{v} & \prod_{i \in I} \mathbb{A}_i & \xrightarrow{\pi_j} & \mathbb{A}_j \\ & & \downarrow n & & \\ & & \prod_{\mathcal{U}} \mathbb{A}_i = \mathbb{B} & & \end{array}$$

Here v is a valuation in the product $\prod_{i \in I} \mathbb{A}_i$, n is the natural quotient map onto the ultraproduct $\mathbb{B}$, and π_j is the j-th coordinate projection. Since n and π_j are surjective, any valuation in $\mathbb{B}$ or $\mathbb{A}_j$ factors through n or π_j respectively. Thus we can compare valuations in $\mathbb{B}$ and $\mathbb{A}_j$ via valuations in $\prod_{i \in I} \mathbb{A}_i$.

Theorem 1. *(Łos's Theorem) Let $\{\mathbb{A}_i \mid i \in I\}$ be a set of $\mathcal{L}$-structures and let $\mathcal{U}$ be an ultrafilter on I. Let $\mathbb{B} = \prod_{\mathcal{U}} \mathbb{A}_i$ be the ultraproduct. If $v\colon X \to \prod_{i \in I} A_i$ is a valuation, then for every formula $\varphi(\bar{x})$ it is the case that*

$$\mathbb{B} \models \varphi[n \circ v] \quad \textit{iff} \quad \{i \in I \mid \mathbb{A}_i \models \varphi[\pi_i \circ v]\} \in \mathcal{U}.$$

Proof. The displayed line in the theorem statement is proved by induction on the complexity of φ, which we may assume is built up from atomic formulas using $\neg, \wedge, \exists$.

Claim 2. *(Terms) For any term t, $t^{\mathbb{B}}[n \circ v] = [\langle t^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I\rangle]_{\theta_{\mathcal{U}}}$.*

- $(t = x_k)$

$$t^{\mathbb{B}}[n \circ v] = x_k[n \circ v] = n \circ v(x_k) = [v(x_k)]_{\theta_{\mathcal{U}}} = [\langle x_k[\pi_i \circ v] \mid i \in I\rangle]_{\theta_{\mathcal{U}}} = [\langle t^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I\rangle]_{\theta_{\mathcal{U}}}$$

- $(t = c)$

$$t^{\mathbb{B}}[n \circ v] = c^{\mathbb{B}}[n \circ v] = c^{\mathbb{B}} = [\langle c^{\mathbb{A}_i} \mid i \in I\rangle]_{\theta_{\mathcal{U}}} = [\langle c^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I\rangle]_{\theta_{\mathcal{U}}} = [\langle t^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I\rangle]_{\theta_{\mathcal{U}}}$$

- $(t = F(t_1, \ldots, t_m))$

$$\begin{aligned} t^{\mathbb{B}}[n \circ v] &= F^{\mathbb{B}}(t_1^{\mathbb{B}}[n \circ v], \ldots, t_m^{\mathbb{B}}[n \circ v]) = F^{\mathbb{B}}([\langle t_1^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I\rangle]_{\theta_{\mathcal{U}}}, \ldots, [\langle t_m^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I\rangle]_{\theta_{\mathcal{U}}}) \\ &= [\langle F^{\mathbb{A}_i}(t_1^{\mathbb{A}_i}[\pi_i \circ v], \ldots, t_m^{\mathbb{A}_i}[\pi_i \circ v]) \mid i \in I\rangle]_{\theta_{\mathcal{U}}} = [\langle t^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I\rangle]_{\theta_{\mathcal{U}}} \end{aligned}$$

Claim 3. *(Atomic formulas)*

- $(s = t)$

$$\begin{aligned} \mathbb{B} \models (s = t)[n \circ v] &\leftrightarrow s^{\mathbb{B}}[n \circ v] = t^{\mathbb{B}}[n \circ v] \\ &\leftrightarrow [\langle s^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I\rangle]_{\theta_{\mathcal{U}}} = [\langle t^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I\rangle]_{\theta_{\mathcal{U}}} \\ &\leftrightarrow \{i \in I \mid s^{\mathbb{A}_i}[\pi_i \circ v] = t^{\mathbb{A}_i}[\pi_i \circ v]\} \in \mathcal{U} \\ &\leftrightarrow \{i \in I \mid \mathbb{A}_i \models (s = t)[\pi_i \circ v]\} \in \mathcal{U} \end{aligned}$$

- $(R(t_1,\dots,t_m))$

$$\begin{aligned}\mathbb{B}\models R(t_1,\dots,t_m)[n\circ v] &\leftrightarrow (t_1^{\mathbb{B}}[n\circ v],\dots,t_m^{\mathbb{B}}[n\circ v])\in R^{\mathbb{B}}\\ &\overset{\text{def}}{\leftrightarrow} \{i\in I\mid (t_1^{\mathbb{A}_i}[\pi_i\circ v],\dots,t_m^{\mathbb{A}_i}[\pi_i\circ v])\in R^{\mathbb{A}_i}\}\in\mathcal{U}\\ &\leftrightarrow \{i\in I\mid \mathbf{A}_i\models R(t_1,\dots,t_m)[\pi_i\circ v]\}\in\mathcal{U}\end{aligned}$$

Claim 4. *(Connectives)*

- $(\neg)$

$$\begin{aligned}\mathbb{B}\models\neg\varphi[n\circ v] &\leftrightarrow \mathbb{B}\not\models\varphi[n\circ v]\\ &\leftrightarrow \{i\in I\mid \mathbb{A}_i\models\varphi[\pi_i\circ v]\}\notin\mathcal{U}\\ &\leftrightarrow I\setminus\{i\in I\mid \mathbb{A}_i\models\varphi[\pi_i\circ v]\}\in\mathcal{U}\\ &\leftrightarrow \{i\in I\mid \mathbb{A}_i\models\neg\varphi[\pi_i\circ v]\}\in\mathcal{U}\end{aligned}$$

- $(\wedge)$

$$\begin{aligned}\mathbb{B}\models(\chi\wedge\varphi)[n\circ v] &\leftrightarrow \mathbb{B}\models\chi[n\circ v]\text{ and }\mathbb{B}\models\varphi[n\circ v]\\ &\leftrightarrow \{i\in I\mid \mathbb{A}_i\models\chi[\pi_i\circ v]\}\in\mathcal{U}\text{ and }\{i\in I\mid \mathbb{A}_i\models\varphi[\pi_i\circ v]\}\in\mathcal{U}\\ &\leftrightarrow \{i\in I\mid \mathbb{A}_i\models\chi[\pi_i\circ v]\}\cap\{i\in I\mid \mathbb{A}_i\models\varphi[\pi_i\circ v]\}\in\mathcal{U}\\ &\leftrightarrow \{i\in I\mid \mathbb{A}_i\models(\chi\wedge\varphi)[\pi_i\circ v]\}\in\mathcal{U}\end{aligned}$$

Claim 5. *$(\exists)$*

[$\Rightarrow$]

$$\begin{aligned}\mathbb{B}\models\exists x_k\varphi[n\circ v] &\longrightarrow \text{there is a valuation } v'\equiv_k v\text{ such that }\mathbb{B}\models\varphi[n\circ v']\\ &\longrightarrow \{i\in I\mid \mathbb{A}_i\models\varphi[\pi_i\circ v']\}\in\mathcal{U}\\ &\longrightarrow \{i\in I\mid \mathbb{A}_i\models\exists x_k\varphi[\pi_i\circ v]\}\in\mathcal{U}\ \ (\text{since } \pi_i\circ v\equiv_k\pi_i\circ v')\end{aligned}$$

[$\Leftarrow$] Assume that $\{i\in I\mid \mathbb{A}_i\models\exists x_k\varphi[\pi_i\circ v]\}=U\in\mathcal{U}$. For each $i\in U$ pick a valuation $w_i\equiv_k\pi_i\circ v$ such that $\mathbb{A}_i\models\varphi[w_i]$. Choose any valuation $v'\colon X\to\prod\mathbb{A}_i$ such that $v'\equiv_k v$ and $\pi_i\circ v'=w_i$ when $i\in U$. Then $\{i\in I\mid \mathbb{A}_i\models\varphi[\pi_i\circ v']\}$ contains U, so $\mathbb{B}\models\varphi[n\circ v']$ by induction, so $\mathbb{B}\models\exists x_k\varphi[n\circ v]$. $\square$