

## Formulas.

In this note we say what is meant by a formal mathematical statement. We first begin by specifying a *language* (called  $L$ ), by which we mean specifying which predicate symbols ( $\mathcal{P} = \{=, <, \dots\}$ ), which operation symbols ( $\mathcal{O} = \{+, \cdot, -, \dots\}$ ), and which constant symbols ( $\mathcal{C} = \{0, 1, \pi, \dots\}$ ) we need for the ideas we want to express.

- Example 1.** (1) The language of *set theory* has one predicate symbol  $\in$ , no operation symbols, and no constant symbols.  
 (2) One language for *number theory* (i.e., the theory of the natural numbers) has one operation symbol,  $S$  (for successor), one constant symbol,  $0$  (for zero), and no non-logical predicate symbols.  
 (3) One language for the real numbers has operation symbols  $\mathcal{O} = \{+, \cdot, -\}$ , constant symbols  $\mathcal{C} = \{0, 1\}$ , and predicate symbols  $\mathcal{P} = \{<\}$ .

Fixing  $L$ , we can define terms, atomic formulas, then arbitrary formulas in this language.

**Definition 2.** The set of all  $L$ -terms is the smallest set  $\mathcal{T}$  such that

- (i)  $\mathcal{T}$  contains all variables and constant symbols, and
- (ii) if  $f \in \mathcal{O}$  is an  $n$ -ary operation symbol and  $t_1, \dots, t_n \in \mathcal{T}$ , then  $f(t_1, \dots, t_n) \in \mathcal{T}$ .

- Example 3.** (1) In the language of set theory the only terms are variables.  
 (2) In the language of number theory whose nonlogical symbols are  $0$  and  $S$ , the only terms are of the form  $S^k(0)$  and  $S^k(x_i)$ ,  $k = 0, 1, 2, \dots$ .  
 (3) In the language of the real numbers whose nonlogical symbols are  $+, \cdot, -, 0, 1, <$  there are very complicated terms like  $((x_1 \cdot x_{17}) + ((x_1 \cdot 0) \cdot x_9)) + 1$ .

**Definition 4.** The set of all *atomic  $L$ -formulas* is the set of all strings  $P(t_1, \dots, t_n)$  where  $P$  is an  $n$  variable predicate symbol and the  $t_i$  are terms.

- Example 5.** (1) In the language of set theory the only atomic formulas are of the form  $(x_i \in x_j)$ .  
 (2) In the language of number theory whose nonlogical symbols are  $0$  and  $S$ , the only atomic formulas are equations of the form  $(S^k(x_i) = S^\ell(x_j))$ ,  $(S^k(x_i) = S^\ell(0))$ ,  $(S^k(0) = S^\ell(x_j))$ , and  $(S^k(0) = S^\ell(0))$ .  
 (3) In the language of the real numbers whose nonlogical symbols are  $+, \cdot, -, 0, 1, <$  there are very complicated atomic formulas, including  $(1 < (x \cdot x))$  or  $((x_1 + (x_2 + x_3)) = ((x_1 + x_2) + x_3))$ .

**Definition 6.** The set of all  $L$ -formulas is the smallest set  $\mathcal{F}$  such that

- (i)  $\mathcal{F}$  contains all atomic formulas, and
- (ii) if  $\alpha, \beta \in \mathcal{F}$  and  $x$  is a variable, then the following are in  $\mathcal{F}$ :  $(\alpha \wedge \beta)$ ,  $(\alpha \vee \beta)$ ,  $(\alpha \rightarrow \beta)$ ,  $(\alpha \leftrightarrow \beta)$ ,  $(\neg \alpha)$ ,  $(\forall x \alpha)$ ,  $(\exists x \alpha)$ .

**Example 7.** In any language, the formulas get complicated. Here are some examples.

- (1) (Set theory) We can express “ $x$  is a subset of  $y$ ” with the formula  $\alpha(x, y) = “\forall z ((z \in x) \rightarrow (z \in y))”$ .
- (2) (Number theory) We can express that the successor function is 1-1 with the formula  $\beta = \forall x \forall y ((S(x) = S(y)) \rightarrow (x = y))$ .
- (3) (Real numbers) We can express that any monic cubic polynomial has a root with the formula  $\gamma = \forall y_1 \forall y_2 \forall y_3 \exists x (x^3 + y_1 \cdot x^2 + y_2 \cdot x + y_3 = 0)$ .

**Exercises.** Express the given fact or relation in the language whose nonlogical symbols are those given.

- (1) Express “There is a set with no elements” in the language of set theory.
- (2) Express “ $x$  has exactly two elements” in the language of set theory.
- (3) Write the Axiom of Extensionality in the language of set theory.
- (4) One language for ordered sets has  $\leq$  as its only nonlogical symbol. In this language express “ $x$  is not the largest element and not the smallest element.”
- (5) Express Fermat’s Last Theorem in a language for number theory whose nonlogical symbols are  $0, +, \cdot, \wedge, <$ . (Fermat’s Last Theorem is the statement that if  $x, y, z, n$  are nonzero natural numbers and  $n$  is at least 3, then  $x^n + y^n = z^n$  does not hold.)