The Compactness Theorem.

Compactness Theorem. *If Σ is a finitely satisfiable set of sentences, then Σ is satisfiable.*

The first stage of the proof is to enlarge Σ to a finitely satisfiable set of sentences having additional properties.

If Λ is a directed set, then a Λ-*directed family* is a Λ-indexed family of sets, $(A_\lambda)_{\lambda\in\Lambda}$, such that $\mu < \nu$ implies $A_\mu \subseteq A_\nu$.

Lemma 1. *Let Λ is a directed set, let $(\mathcal{L}_\lambda)_{\lambda\in\Lambda}$ be Λ-directed family of languages, and let $(\Sigma_\lambda)_{\lambda\in\Lambda}$ be a Λ-directed family of sets of sentences with $\Sigma_\lambda \subseteq \mathcal{L}_\lambda$. If each Σ_λ is finitely satisfiable, then $\bigcup_{\lambda\in\Lambda}\Sigma_\lambda$ is a finitely satisfiable set of $\bigcup_{\lambda\in\Lambda}\mathcal{L}_\lambda$-sentences.*

Proof. Any finite subset Σ_0 of $\bigcup_{\lambda\in\Lambda}\Sigma_\lambda$ lies in some Σ_λ, since Λ is directed. Σ_0 can be satisfied by some $\mathcal{L}_\lambda$-structure, $\mathbf{A}$, since Σ_λ is finitely satisfiable and all symbols of Σ_λ belong to $\mathcal{L}_\lambda$. An arbitrary expansion of $\mathbf{A}$ to the language $\bigcup_{\lambda\in\Lambda}\mathcal{L}_\lambda$ with still satisfy Σ_0. $\square$

Lemma 2. *If Σ is a finitely satisfiable set of $\mathcal{L}$-sentences, then there is a superset $\Sigma' \supseteq \Sigma$ of $\mathcal{L}$-sentences that*

(i) *is finitely satisfiable, and*

(ii) *satisfies the **completeness condition** for $\mathcal{L}$, which is the statement that $\sigma\in\Sigma'$ or $(\neg\sigma)\in\Sigma'$ for every $\mathcal{L}$-sentence σ.*

Proof. Let $\kappa = |\mathcal{L}|$ and let $(\sigma_\lambda)_{\lambda<\kappa}$ be an enumeration of all $\mathcal{L}$-sentences. Define a sequence of sets of $\mathcal{L}$-sentences by

$$\begin{aligned}\Sigma_0 &:= \Sigma\\ \Sigma_{\lambda+1} &:= \begin{cases}\Sigma_\lambda\cup\{\sigma_\lambda\} & \text{if finitely satisfiable;}\\ \Sigma_\lambda\cup\{(\neg\sigma_\lambda)\} & \text{else.}\end{cases}\\ \Sigma_\lambda &:= \textstyle\bigcup_{\mu<\lambda}\Sigma_\mu \qquad\qquad \text{if } \lambda \text{ is limit.}\end{aligned}$$

Σ_0 is finitely satisfiable by assumption.

Suppose that Σ_λ is finitely satisfiable but $\Sigma_{\lambda+1}$ is not. Then neither $\Sigma_\lambda\cup\{\sigma\}$ nor $\Sigma_\lambda\cup\{(\neg\sigma)\}$ is finitely satisfiable, meaning that there exists a finite (satisfiable) subset $\Gamma\subseteq\Sigma$ such that neither $\Gamma\cup\{\sigma\}$ nor $\Gamma\cup\{(\neg\sigma)\}$ is satisfiable. If $\mathbf{A}$ is a model of Γ, we are forced to conclude that neither σ nor $(\neg\sigma)$ holds in $\mathbf{A}$, which is impossible.

Finally, if Σ_μ is finitely satisfiable for all $\mu<\lambda$, λ limit, then Σ_λ is finitely satisfiable by Lemma 1.

This proves that $\Sigma' := \Sigma_\kappa$ is finitely satisfiable. Since each $\mathcal{L}$-sentence is some σ_λ, and Σ' $(\supseteq \Sigma_{\lambda+1})$ contains either σ_λ or $(\neg\sigma_\lambda)$, Σ' satisfies the completeness condition for $\mathcal{L}$. $\square$

Lemma 3. *If Σ is a finitely satisfiable set of $\mathcal{L}$-sentences, then there are a set C of new constants, an extended language $\mathcal{L}(C) = \mathcal{L}\cup C$, and a superset $\Sigma'\supseteq\Sigma$ consisting of $\mathcal{L}(C)$-sentences that*

(i) *is finitely satisfiable, and*
(ii) ***has witnesses for $\mathcal{L}$-formulas****, which means that for every $\mathcal{L}$-formula $\alpha(x)$ in one free variable there is a constant symbol c such that $((\exists x\alpha(x)) \to \alpha(c)) \in \Sigma'$.*

Proof. Let $\kappa = |\mathcal{L}|$ and let $(\alpha_\lambda(x))_{\lambda<\kappa}$ be an enumeration of all $\mathcal{L}$-formulas with one free variable. Let C be a set of κ-many new constant symbols. Define a sequence of sets of $\mathcal{L}(C)$-sentences by

$$\begin{aligned} \Sigma_0 &:= \Sigma \\ \Sigma_{\lambda+1} &:= \Sigma_\lambda \cup \{((\exists x\alpha_\lambda(x)) \to \alpha(c_\lambda))\} \\ \Sigma_\lambda &:= \textstyle\bigcup_{\mu<\lambda} \Sigma_\mu \qquad \text{if } \lambda \text{ is limit.} \end{aligned}$$

Suppose that Σ_λ is finitely satisfiable but $\Sigma_{\lambda+1}$ is not. Let $\Gamma \subseteq \Sigma_\lambda$ be a finite set such that $\Gamma \cup \{((\exists x\alpha_\lambda(x)) \to \alpha(c_\lambda))\}$ is unsatisfiable. Γ itself is satisfiable, so there must be a structure $\mathbf{A}$ in an appropriate language that satisfies Γ; we may assume that the language is $\mathcal{L}(C_\lambda)$ where $C_\lambda = \{c_\mu \mid \mu < \lambda\}$. We claim that $\mathbf{A}$ can be modified to an $\mathcal{L}(C_{\lambda+1})$-structure satisfying $\Gamma \cup \{((\exists x\alpha_\lambda(x)) \to \alpha(c_\lambda))\}$.

If $\mathbf{A} \models (\exists x\alpha(x))$, $\mathbf{A} \models \alpha[a]$ for some $a \in A$. Define $c_\lambda^{\mathbf{A}} = a$. If $\mathbf{A} \not\models (\exists x\alpha(x))$, then define $c_\lambda^{\mathbf{A}} \in A$ arbitrarily. By interpreting c_λ in A we create a $\mathcal{L}(C_{\lambda+1})$-structure $\mathbf{A}_c := \langle \mathbf{A}; c_\lambda\rangle$. This structure still satisfies Γ, since c_λ does not appear in any sentence in Γ, but it now also satisfies $((\exists x\alpha_\lambda(x)) \to \alpha(c_\lambda))$. This contradicts our earlier supposition that $\Gamma \cup \{((\exists x\alpha_\lambda(x)) \to \alpha(c_\lambda))\}$ is unsatisfiable. □

Corollary 4. *If Σ is a finitely satisfiable set of $\mathcal{L}$-sentences, then there are a set C of new constants, an extended language $\mathcal{L}(C) = \mathcal{L} \cup C$, and a superset $\Sigma' \supseteq \Sigma$ consisting of $\mathcal{L}(C)$-sentences that*
(i) *is finitely satisfiable, and*
(ii) *satisfies the completeness condition for $\mathcal{L}(C)$, and*
(iii) *has witnesses for $\mathcal{L}(C)$-formulas.*

Proof. Starting with Σ, we can use Lemmas 2 and 3 to create an increasing chain of finitely satisfiable sets $\Sigma \subseteq \Sigma_c \subseteq \Sigma_{cw} \subseteq \Sigma_{cwc} \subseteq \Sigma_{cwcw} \subseteq \cdots$ in increasing languages $\mathcal{L} = \mathcal{L} \subseteq \mathcal{L}(C_w) = \mathcal{L}(C_w) \subseteq \mathcal{L}(C_{ww}) = \cdots$, where subscript c means "enforce the completeness condition" and subscript w means "add witnesses". The union set Σ' will be finitely satisfiable and, with respect to the union language $\mathcal{L}'$, will satisfy the completeness condition and have witnesses. □

Proof of the Compactness Theorem (really the second stage). Call a set Σ of $\mathcal{L}$-sentences a ***Henkin set*** if it is finitely satisfiable, satisfies the completeness condition for $\mathcal{L}$, and has witnesses for $\mathcal{L}$-formulas. We will prove the Compactness Theorem in the special case where Σ is a Henkin set, then deduce that the theorem holds in general.

We proceed to construct a model of Σ under the assumption that it is a Henkin set.

Step I. (Defining the universe.)

Let C be the set of constant symbols of $\mathcal{L}$. Define a relation $\sim$ on C by: $c \sim d$ iff $(c \doteq d) \in \Sigma$.

Claim 5. *$\sim$ is an equivalence relation on C.*

Proof of claim. If $\sim$ were not reflexive on C, then there would be a $c \in C$ such that $(c \doteq c) \notin \Sigma$. The completeness condition for Σ forces $(\neg(c \doteq c)) \in \Sigma$. By the finite satisfiability of Σ there must be an $\mathcal{L}$-structure $\mathbf{X}$ satisfying $(\neg(c \doteq c))$. But this is impossible, since c interprets as an element $c^{\mathbf{X}} \in X$, $\doteq$ interprets as equality on X, and $c^{\mathbf{X}} \neq c^{\mathbf{X}}$ never holds.

Similarly, if $\sim$ failed to be symmetric or transitive on C, then applying the completeness condition and the finite satisfiability of Σ we obtain a contradiction. $\square$

Write $\overline{c}$ for $c/\sim$ and $\overline{C}$ for $C/\sim$. The universe of our model will be $A := \overline{C}$.

Step II. (Interpreting the $\mathcal{L}$-symbols.)

We will interpret the nonlogical symbols of $\mathcal{L}$ according to the following rules.

(R) If R is an n-ary relation symbol, then define

$$(\overline{c}_0, \dots, \overline{c}_{n-1}) \in R^{\mathbf{A}} \quad \text{iff} \quad R(c_0, \dots, c_{n-1}) \in \Sigma.$$

(F) If F is an n-ary function symbol, then define

$$F^{\mathbf{A}}(\overline{c}_0, \dots, \overline{c}_{n-1}) = \overline{c}_n \quad \text{iff} \quad (F(c_0, \dots, c_{n-1}) \doteq c_n) \in \Sigma.$$

(c) If c is a constant symbol, then define $c^{\mathbf{A}} = \overline{c}$.

To eliminate any concerns about the validity of these definitions we prove the following

Claim 6. (1) *For every term t with no variables there is an element $c \in C$ such that $(t \doteq c) \in \Sigma$; moreover, $t^{\mathbf{A}} = c^{\mathbf{A}}$.*

(2) *If for all subscripts i we have $c_i \in C$, t_i is a term with no variables, and $(t_i \doteq c_i) \in \Sigma$, then $R(t_0, \dots, t_{n-1}) \in \Sigma$ iff $R(c_0, \dots, c_{n-1}) \in \Sigma$.*

(3) *If for all subscripts i we have $c_i \in C$, t_i is a term with no variables, and $(t_i \doteq c_i) \in \Sigma$, then $(F(t_0, \dots, t_{n-1}) \doteq t_n) \in \Sigma$ iff $(F(c_0, \dots, c_{n-1}) \doteq c_n) \in \Sigma$.*

Proof of claim. If t is a constant, the first part of Item (1) follows from the fact that $\sim$ is reflexive. The second part follows from the fact that the equality relation on A is reflexive. Assume that $t = F(t_0, \dots, t_{n-1})$ and that Item (1) is true for each t_i. Then there are constants c_i such that $(t_i \doteq c_i) \in \Sigma$ and $t_i^{\mathbf{A}} = c_i^{\mathbf{A}}$. For the formula $\alpha(x) := (F(c_0, \dots, c_{n-1}) \doteq x)$ there is a constant symbol c such that $((\exists x \alpha(x)) \to \alpha(c)) \in \Sigma$, since Σ has witnesses. The formula $(\exists x \alpha(x))$ is valid, since F is an function symbol, so $(\exists x \alpha(x)) \in \Sigma$. It follows that $\alpha(c) \in \Sigma$, else by completeness Σ contains the unsatisfiable finite set

$$\{(\exists x \alpha(x)), ((\exists x \alpha(x)) \to \alpha(c)), (\neg \alpha(c))\}.$$

This shows that (i) $(F(c_0, \dots, c_{n-1}) \doteq c) \in \Sigma$, hence (ii) $F^{\mathbf{A}}(c_0^{\mathbf{A}}, \dots, c_{n-1}^{\mathbf{A}}) = c^{\mathbf{A}}$.

Regarding (i), the set

$$\{(t_0 \doteq c_0), \dots, (t_{n-1} \doteq c_{n-1}), (F(c_0, \dots, c_{n-1}) \doteq c), (\neg(F(t_0, \dots, t_{n-1}) \doteq c))\}$$

is unsatisfiable, and each sentence but the last belongs to Σ. By the completeness and finite satisfiability of Σ we have $(F(t_0, \dots, t_{n-1}) \doteq c) \in \Sigma$ (or $(t \doteq c) \in \Sigma$), as desired.

Regarding (ii), we have $t^{\mathbf{A}} = F^{\mathbf{A}}(t_0^{\mathbf{A}}, \ldots, t_{n-1}^{\mathbf{A}}) = F^{\mathbf{A}}(c_0^{\mathbf{A}}, \ldots, c_{n-1}^{\mathbf{A}}) = c^{\mathbf{A}}$. This completes the proof of Item (1).

If Item (2) were false, then by completeness Σ would contain either

$$\{(t_0 \doteq c_0), \ldots, (t_{n-1} \doteq c_{n-1}), R(t_0, \ldots, t_{n-1}), (\neg R(c_0, \ldots, c_{n-1}))\}$$

or

$$\{(t_0 \doteq c_0), \ldots, (t_{n-1} \doteq c_{n-1}), (\neg R(t_0, \ldots, t_{n-1}))R(c_0, \ldots, c_{n-1}), \}.$$

Both are unsatisfiable, so (2) is not false. A similar argument proves (3). $\square$

Step III. (The $\mathcal{L}$-structure $\mathbf{A} := \langle A; R, \ldots, F, \ldots, c, \ldots\rangle$ satisfies Σ.)

Claim 7. *If σ is an $\mathcal{L}$-sentence, then $\mathbf{A} \models \sigma$ iff $\sigma \in \Sigma$.*

Proof of claim. Assume first that σ is $R(t_0, \ldots, t_{n-1})$ for some R, n and t_i. Since this is an atomic sentence, the t_i have no variables, hence Claim 6 (1) guarantees that there are constant symbols c_i such that $(t_i \doteq c_i) \in \Sigma$ and $t_i^{\mathbf{A}} = c_i^{\mathbf{A}}$. We have the following equivalences:

$$\begin{array}{rl}
\mathbf{A} \models \sigma & \text{iff } \mathbf{A} \models R(t_0, \ldots, t_{n-1}) \\
& \text{iff } (t_0^{\mathbf{A}}, \ldots, t_{n-1}^{\mathbf{A}}) \in R^{\mathbf{A}} \\
& \text{iff } (c_0^{\mathbf{A}}, \ldots, c_{n-1}^{\mathbf{A}}) \in R^{\mathbf{A}} \\
& \text{iff } R(c_0, \ldots, c_{n-1}) \in \Sigma \\
& \text{iff } R(t_0, \ldots, t_{n-1}) \in \Sigma \\
& \text{iff } \sigma \in \Sigma.
\end{array}$$

Each step follows from the definitions or from Claim 6.

The case where σ is $(F(t_0, \ldots, t_{n-1}) \doteq t_n)$ for some F, n and t_i is similar.

Now assume that $\sigma = (\alpha \wedge \beta)$. We have the following equivalences:

$$\begin{array}{rl}
\mathbf{A} \models (\alpha \wedge \beta) & \text{iff } \mathbf{A} \models \alpha \text{ and } \mathbf{A} \models \beta \\
& \text{iff } \alpha \in \Sigma \text{ and } \beta \in \Sigma \\
& \text{iff } (\alpha \wedge \beta) \in \Sigma.
\end{array}$$

The only step requiring justification is the equivalence of the last two lines. If we had $\alpha, \beta \in \Sigma$ and $(\alpha\wedge\beta) \notin \Sigma$, then the completeness condition yields that $\{\alpha, \beta, (\neg(\alpha\wedge\beta))\} \subseteq \Sigma$, contrary to the finite satisfiability of Σ. In the reverse direction, if $(\alpha \wedge \beta) \in \Sigma$ and, say, $\alpha \notin \Sigma$, then $\{(\neg\alpha), (\alpha \wedge \beta)\} \subseteq \Sigma$ contradicts the finite satisfiability of Σ.

The case where $\sigma = (\neg\alpha)$ can be argued similarly.

Finally we assume that $\sigma = (\exists x\alpha(x))$. We have the following equivalences:

$$\begin{array}{rl}
\mathbf{A} \models (\exists x\alpha(x)) & \text{iff } \mathbf{A} \models \alpha[a] \text{ for some } a \in A \\
& \text{iff } \mathbf{A} \models \alpha(c) \text{ for some } c \in C \\
& \text{iff } \alpha(c) \in \Sigma \text{ for some } c \in C \\
& \text{iff } (\exists x\alpha(x)) \in \Sigma.
\end{array}$$

We need to defend the claim that the last two lines are equivalent. The claim that $\alpha(c) \in \Sigma$ for some $c \in C$ implies $(\exists x\alpha(x)) \in \Sigma$ follows from the completeness condition and the finite satisfiability of Σ. The claim that $(\exists x \alpha(x)) \in \Sigma$ implies $\alpha(c) \in \Sigma$ for some $c \in C$ follows from finite satisfiability and the fact that Σ has witnesses for $\mathcal{L}$-formulas. □

Step IV. (The theorem for general Σ follows from the theorem for Σ a Henkin set.)

Let Σ be a finitely satisfiable set of $\mathcal{L}$-sentences. According to Corollary 4 there is a language $\mathcal{L}' \supseteq \mathcal{L}$ and a Henkin set $\Sigma' \supseteq \Sigma$ in the language $\mathcal{L}'$. We have shown that Σ' has a model, $\mathbf{A}$. The reduct of $\mathbf{A}$ to the symbols in $\mathcal{L}$ is a model of Σ. □

Corollary 8. *An unsatisfiable set of sentences contains a finite unsatisfiable subset.*

Proof. This is the contrapositive of the Compactness Theorem. □

Corollary 9. *If $\Sigma \models \sigma$, then $\Sigma_0 \models \sigma$ for some finite subset $\Sigma_0 \subseteq \Sigma$.*

Proof. Rewrite Corollary 8 using the equivalence of "$\Sigma \models \sigma$" with "$\Sigma \cup \{(\neg\sigma)\}$ is unsatisfiable" (which follows from the definition of $\models$). □

Corollary 10. *If Σ has a model, then Σ has a model of size at most $|\Sigma| + \omega$.*

Proof. First, there is no harm in assuming that the only nonlogical symbols of $\mathcal{L}$ are those that appear in Σ, so in particular $|\mathcal{L}| = |\Sigma| + \omega$. Next, show that the number of constant symbols added in the course of the proof is at most $|\mathcal{L}|$. Then $|A| = |C/\sim| \leq |C| \leq |\mathcal{L}|$. □

The case of Corollary 10 where $\Sigma = \{\sigma\}$ (and hence $|\Sigma| + \omega = \omega$) is called *Löwenheim's Theorem.*

Corollary 11. *If Σ has arbitrarily large finite models, then Σ has an infinite model.*
If Σ has an infinite model, then Σ has arbitrarily large infinite models.

Proof. Expand the language to contain some infinite set C of new constant symbols, say $\{c_\lambda \mid \lambda < \kappa\}$ where κ is an infinite cardinal. Let $\Gamma = \{(\neg(c_\mu \doteq c_\nu)) \mid \mu < \nu < \kappa\}$. If Σ has arbitrarily large finite models or an infinite model, then $\Sigma \cup \Gamma$ is finitely satisfiable and is therefore satisfiable. Any model has cardinality at least κ. □

Corollary 12. *If $\Sigma \cup \{\sigma\}$ is a set of $\mathcal{L}$-sentences and σ holds in every infinite model of Σ, then there is a finite N such that σ holds in every model of Σ of size greater than N.*

Proof. If this is not true, then $\Sigma \cup \{(\neg\sigma)\}$ has arbitrarily large finite models and no infinite models, contrary to Corollary 11. □

Corollary 13. *The following conditions are equivalent for any class $\mathcal{K}$ of $\mathcal{L}$-structures.*

(1) *$\mathcal{K}$ is finitely axiomatizable.*
(2) *$\mathcal{K}$ is axiomatizable by a single $\mathcal{L}$-sentence.*
(3) *Both $\mathcal{K}$ and its complement are axiomatizable.*

Proof. [(1)⇒(2)] Replace a finite set of axioms for $\mathcal{K}$ with its conjunction.

[(2)⇒(3)] If σ axiomatizes $\mathcal{K}$, then $(\neg\sigma)$ axiomatizes the complement of $\mathcal{K}$.

[(3)⇒(1)] Suppose that Σ axiomatizes $\mathcal{K}$ and Γ axiomatizes its complement. Then $\Sigma\cup\Gamma$ is unsatisfiable, so there exist finite sets $\Sigma_0 \subseteq \Sigma$ and $\Gamma_0 \subseteq \Gamma$ such that $\Sigma_0 \cup \Gamma_0$ is unsatisfiable.

I claim that Σ_0 axiomatizes $\mathcal{K}$. Σ_0 axiomatizes a class containing $\mathcal{K}$, since $\Sigma_0 \subseteq \Sigma$. But the containment cannot be proper, else there is some $\mathbf{A} \notin \mathcal{K}$ such that $\mathbf{A} \models \Sigma_0$, and this would lead to $\mathbf{A} \models \Sigma_0 \cup \Gamma_0$, contrary to the conclusion of the last paragraph. □

Corollary 14. *If $T_0 \subset T_1 \subset \cdots$ is a strictly increasing chain of $\mathcal{L}$-theories, then the class of $\mathcal{L}$-structures $\mathbb{A}$ such that $\mathbb{A} \not\models T_i$ for some i is not elementary.*

Proof. The class identified in the corollary is the complement of the class axiomatized by $\bigcup T_i$, so it suffices to show that $\bigcup T_i$ is not finitely axiomatizable. This follows from the fact that the chain $T_0 \subset T_1 \subset \cdots$ is strictly increasing and consists of theories. □

Corollary 15. *If Σ and Γ axiomatize the same class $\mathcal{K}$ and Γ is finite, then there is a finite subset $\Sigma_0 \subseteq \Sigma$ that axiomatizes $\mathcal{K}$.*

Proof. We may assume that Γ consists of a single sentence γ. Now $(\neg\gamma)$ axiomatizes the complement of $\mathcal{K}$, so by the argument for Corollary 13 (3)⇒(1) we get that some finite subset $\Sigma_0 \subseteq \Sigma$ axiomatizes $\mathcal{K}$. □

Corollary 16. *If $\mathbf{A}$ and $\mathbf{B}$ are elementarily equivalent $\mathcal{L}$-structures, then there is a third $\mathcal{L}$-structure $\mathbf{C}$ that is a common elementary extension of each of them.*

Proof. Let $\mathbf{A}_A$ be the expansion of $\mathbf{A}$ by constants and let $\mathbf{B}_B$ be the expansion of $\mathbf{B}$ by constants. Assume that all the new constant symbols are distinct. I claim that $\Sigma := \mathrm{Th}(\mathbf{A}_A)\cup\mathrm{Th}(\mathbf{B}_B)$ is finitely satisfiable. To see this, note that any finite set of sentences from either theory is equivalent to a single one, so any finite set of sentences in Σ is equivalent to $\alpha(\overline{a}) \wedge \beta(\overline{b})$ where (i) $\alpha(\overline{x})$ and $\beta(\overline{x})$ are $\mathcal{L}$-formulas, (ii) the coordinates of $\overline{a}$ are symbols representing elements of A while the coordinates of $\overline{b}$ are symbols representing elements of B, and (iii) $\mathbf{A} \models \alpha(\overline{a})$ and $\mathbf{B} \models \beta(\overline{b})$.

Now $\mathbf{B} \models \exists\overline{y}\beta(\overline{y})$, so we also have $\mathbf{A} \models \exists\overline{y}\beta(\overline{y})$. Choose a tuple $\overline{a}'$ such that $\mathbf{A} \models \beta(\overline{a}')$. Interpreting the symbols in $\overline{b}$ in $\mathbf{A}$ as $\overline{a}'$ we get from $\mathbf{A}$ a structure in the language $\mathcal{L}\cup\{\overline{a},\overline{b}\}$ satisfying $\alpha(\overline{a}) \wedge \beta(\overline{b})$, verifying the finite satisfiability of Σ.

Let $\mathbf{C}$ be a model of Σ. Then $\mathbf{C}$ is a model of $\mathrm{Th}(\mathbf{A}_A)$, making $\mathbf{C}$ and elementary extension of $\mathbf{A}$, similarly of $\mathbf{B}$. □

Corollary 17. *If $\mathbf{A}$ is an $\mathcal{L}$-structure, then any set of of types of Th$(\mathbf{A})$ can be realized in some elementary extension of $\mathbf{A}$.*

Applications. (Terminology. The following are synonymous with each other: "finitely axiomatizable class", "strictly elementary class", "basic elementary class", and "EC-class". Any class that is the intersection of such classes is an "axiomatizable class", an "elementary class", or an "EC_Δ- (or EC_δ)-class". The symbols Δ/δ refer to "Durchschnitt".)

(1) The class of finite sets (or finite $\mathcal{L}$-structures) is not elementary.
(2) The class of infinite sets is elementary, but not finitely axiomatizable.
(3) (Robinson's Principle) Let $\Sigma \cup \{\sigma\}$ be a set of sentences in the language of fields. Assume that Σ contains the field axioms. Then σ is true in all models of Γ of characteristic zero iff σ is true in all models of Γ of characteristic p for sufficuently large p.
(4) (Ax-Grothendieck Theorem) If $V \subseteq \mathbb{C}^n$ is an algebraic variety, then any regular map $\mathbf{m}\colon V \to V$ that is injective is also surjective.
(5) The class of simple groups is not elementary.
(6) The class of torsion groups is not elementary.
(7) If G is a group with elements of arbitrarily large finite order, then G is elementarily equivalent to a group H that has an element of infinite order.
(8) There is no formula $\tau(x)$ in the language of groups that defines the set of torsion elements of a group. (I.e., no $\tau(x)$ such that, for every group G, $G \models \tau[g]$ iff g is a torsion element.)
(9) The class of torsion free groups is elementary, but not finitely axiomatizable.
(10) The class of algebraically closed fields of characteristic zero is axiomatizable, but not finitely axiomatizable.
(11) ($\exists$ nonstandard models of $\mathrm{Th}(\mathbb{R})$) The field of real numbers is elementarily equivalent to an ordered field $\mathbb{F}$ that contains a positive infinitesimal element.
(12) The property of $\mathbb{R}$ that every nonempty subset with an upper bound has a least upper bound cannot be expressed by any set of first-order sentences.
(13) ($\exists$ nonstandard models of $\mathrm{Th}(\mathbb{N})$) The structure $\langle \mathbb{N}; \cdot, + \rangle$ is elementarily equivalent to a structure with infinitely large elements (in fact, infinitely large prime numbers).
(14) There are $2^{\aleph_0}$ nonisomorphic countable structures elementarily equivalent to $\langle \mathbb{N}; \cdot, + \rangle$.
(15) The class of well-ordered sets is not elementary.
(16) Every partial order can be extended to a linear order.
(17) The class of connected graphs is not elementary.
(18) There is no first-order formula $\gamma(x, y)$ in the language of graphs that defines the relation "there is a path from x to y".
(19) The class of acyclic graphs is elementary, but not finitely axiomatizable.
(20) (Erdös) A graph is k-colorable iff each finite subgraph is k-colorable.
(21) (Kőnig's Lemma) Any infinite, finitely branching tree has an infinite path.