MODEL THEORY
HOMEWORK ASSIGNMENT IV

Read Chapter 4.

PROBLEMS

1. (Jack, Lotfi) The theory T of countably many independent unary relations is the theory in the language with relation symbols $R_n(x)$, $n < \omega$, which contains all sentences of the form

$$\exists x(R_{i_1}(x) \wedge \cdots \wedge R_{i_m}(x) \wedge \neg R_{j_1}(x) \wedge \cdots \wedge \neg R_{j_n}(x))$$

whenever $i_1, \ldots, i_m, j_1, \ldots, j_n$ are distinct.

(a) Show that T has quantifier elimination and is complete.

(b) Derive from (a) that any n-type is generated by formulas of the form $\pm$atomic, i.e., those of the form $x_i = x_j$, $x_i \neq x_j$, $R_i(x_j)$, and $\neg R_i(x_j)$.

(c) Explain why $S_n(T)$ has no isolated points, hence is homeomorphic to the Cantor set.

2. (Sparks, Tanksalvala) Show that if $\mathbb{A}$ is atomic and $U \subseteq A$ is a finite subset, then $\mathbb{A}_U$ is atomic.

3. (Jack, Lotfi) A structure is *ultrahomogeneous* if every isomorphism between finitely generated substructures extends to an automorphism. It is known that if $\mathbb{A}$ is a finite $\mathcal{L}$-structure, then $\mathrm{Th}(\mathbb{A})$ has quantifier elimination iff $\mathbb{A}$ is ultrahomogeneous. (This can be proved using the theorem discussed in class on March 18, but here you may assume it.)

In this problem you are to prove that a finite abelian group is ultrahomogeneous iff every Sylow subgroup is a direct power of a cyclic group.

4. (Sparks, Tanksalvala) Let T be a theory in a countable language. Suppose that $p = p(x)$ and $q = q(y)$ are partial 1-types. Show that the following condition is sufficient to guarantee that T has a model realizing p and omitting q:

For every formula $\varphi(x, y)$ there is a formula $\beta(y) \in q$ such that for any $\alpha_1(x), \ldots, \alpha_n(x) \in p$, if

$$T \cup \{\exists x \exists y(\varphi(x, y) \wedge \alpha_1(x) \wedge \cdots \wedge \alpha_n(x))\}$$

is satisfiable, then

$$T \cup \{\exists x \exists y(\varphi(x,y) \wedge \alpha_1(x) \wedge \cdots \wedge \alpha_n(x) \wedge \neg\beta(y))\}$$

is satisfiable. (Hint: Show that q is unsupported as a partial type for $T \cup p(c)$ where c is a new constant.)