MODEL THEORY
HOMEWORK ASSIGNMENT II

Read Chapter 2.

PROBLEMS

1. (Berg, Lessard, Shriner) An ultrafilter is *uniform* if all of its sets have the same size. Show that a regular ultrafilter is uniform.

2. (Jack, Sparks) Let G be the infinite (connected) graph whose vertex set is $\mathbb{Z}$ and whose adjacency relation relates each integer only to its immediate successor and its immediate predecessor. Let $2G$ denote the graph that is equal to two disjoint copies of G. Show that G and $2G$ have isomorphic ultrapowers. Deduce that "connectivity" is not a first-order expressible property of graphs.

3. (Lotfi, Tanksalvala) Show that any structure is embeddable in an ultraproduct of its finitely generated substructures. Conclude that any universal class is generated by its finitely generated members. Show that this statement about universal classes is not true for arbitrary elementary classes. (Universal class = a class axiomatizable by universally quantified sentences.)

4. (Berg, Lessard, Shriner) Two unrelated problems:

(a) Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. Show that an ultraproduct of ultraproducts of members of $\mathcal{K}$ is isomorphic to an ultraproduct of members of $\mathcal{K}$.

(b) Let $\mathcal{K}$ be a finite set of finite $\mathcal{L}$-structures. Show that any ultraproduct of members of $\mathcal{K}$ is isomorphic to some member of $\mathcal{K}$.

5. (Jack, Sparks) Look up the definitions of Σ_1^1 and Π_1^1 sentences if necessary, and include the definitions in your solution to the following problems.

(a) Show that ultraproducts preserve Σ_1^1-sentences.

(b) Give an example of a Π_1^1-sentence not preserved by ultraproducts.

6. (Lotfi, Tanksalvala) Show that the following conditions on $\mathbb{A}$ are equivalent.

(a) $\mathbb{A}$ is a model of the theory of the class of finite $\mathcal{L}$-structures.
(b) Every $\mathcal{L}$-sentence true in $\mathbb{A}$ holds in some finite $\mathcal{L}$-structure.
(c) $\mathbb{A}$ is elementarily equivalent to an ultraproduct of finite $\mathcal{L}$-structures.

($\mathbb{A}$ is *pseudofinite* if these hold.)

7. (Berg, Lessard, Shriner)

(a) Let $\mathcal{U}$ be an ultrafilter on a set I, and let $\{\mathbb{A}_i \mid i \in I\}$ be a set of $\mathcal{L}$-structures. Show that $\prod_{\mathcal{U}} \mathrm{Aut}(\mathbb{A}_i)$ is embeddable in $\mathrm{Aut}(\prod_{\mathcal{U}} \mathbb{A}_i)$. (Elements of $\prod_{\mathcal{U}} \mathrm{Aut}(\mathbb{A}_i)$ are called internal automorphisms of $\prod_{\mathcal{U}} \mathbb{A}_i$, while other automorphisms are called external.)
(b) Give an explicit example of an external automorphism.