MODEL THEORY: HOMEWORK 4

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4: Let T be a theory in a countable language. Suppose that $p = p(x)$ and $q = q(x)$ are partial $1 - types$. Show that the following condition is sufficient to guarantee that T has a model realizing p and omitting q:

For every formula $\varphi(x, y)$ there is a formula $\beta(y) \in q$ such that for any $\alpha_1(x), ..., \alpha_n(x) \in p$, if $T \cup \{\exists x \exists y(\varphi(x, y) \wedge \alpha_1(x) \wedge ... \wedge \alpha_n(x))\}$ is satisfiable, then $T \cup \{\exists x \exists y(\varphi(x, y) \wedge \alpha_1 \wedge ... \wedge \alpha_n(x) \wedge \neg\beta(y))\}$.

Solution:

For language $\mathcal{L}$ of the theory T, define a constant $c \notin \mathcal{L}$. $T \cup p(x)$ is satisfiable as an $\mathcal{L}$ theory, because $p(x)$ is a $1-$type. This means that $T \cup p(c)$ is satisfiable as an $\mathcal{L} \cup \{c\}$-theory.

Subclaim: For every $\mathcal{L} \cup \{c\}$ formula $\varphi(y)$ such that $T \cup p(c) \cup \{\varphi(y)\}$ is satisfiable, $T \cup p(c) \cup \{\exists y(\varphi(y) \wedge \neg\beta(y))\}$ is satisfiable for some $\beta(y) \in q(y)$.
Subproof:
Recognize that $T \cup p(c)$ is finitely satisfiable because it is satisfiable, so it suffices to show that, for any finite subset $K \subseteq T \cup p(c)$, $K \cup \{\exists y(\varphi(y) \wedge \neg\beta(y))\}$ is satisfiable.

$T \cup p(c)$ is satisfiable, so $T \cup K \subseteq T \cup p(c)$ is satisfiable. $T \cup K = T \cup (K \setminus T)$ and $K \setminus T$ is of the form $\{\alpha_0(c), ..., \alpha_n(c)\}$ with every $\alpha_i(c) \in p(c)$.

$T \cup p(c) \cup \{\varphi(y)\}$ being satisfiable implies $T \cup p(c) \cup \{\exists y \varphi(y)\}$ is satisfiable. $T \cup (K \setminus T) \cup \{\exists y \varphi(y)\}$ is satisfiable implies $T \cup \{(\bigwedge_{i \in \{0,...,n\}} \alpha_i(c)) \wedge (\exists y(\varphi(y)))\}$ is satisfiable. $\varphi(y)$ is of the form $\varphi(c, y)$, because it is an $\mathcal{L} \cup \{c\}$ formula. This means $T \cup \{(\bigwedge_{i \in \{0,...,n\}} \alpha_i(c)) \wedge (\exists y(\varphi(c, y)))\}$ is satisfiable. This implies $T \cup \{\exists x((\bigwedge_{i \in \{0,...,n\}} \alpha_i(x)) \wedge (\exists y(\varphi(x, y))))\}$ is satisfiable. y does not occur free in any $\alpha_i(x)$, so this implies $T \cup \{\exists x \exists y((\bigwedge_{i \in \{0,...,n\}} \alpha_i(x)) \wedge (\varphi(x, y)))\}$ is satisfiable. Note that this is an $\mathcal{L}$ theory, because the constant c does not occur. Using the assumption, this implies $T \cup \{\exists x \exists y((\bigwedge_{i \in \{0,...,n\}} \alpha_i(x)) \wedge (\varphi(x, y) \wedge \neg\beta(y)))\}$ is satisfiable as a $\mathcal{L}$ theory for some $\beta(y) \in q(y)$. Reintroducing c to the language allows for these sentences to be written as $T \cup \{\exists y((\bigwedge_{i \in \{0,...,n\}} \alpha_i(c)) \wedge (\varphi(y) \wedge \neg\beta(y)))\}$, which is then satisfiable. Thus $T \cup \{\alpha_1(c), ..., \alpha_n(c), (\exists y(\varphi(y) \wedge \neg\beta(y)))\}$ is satisfiable, which equals $T \cup (K \setminus T) \cup \{(\exists y(\varphi(y) \wedge \neg\beta(y))\}$. This being satisfiable implies $K \cup \{(\exists y(\varphi(y) \wedge \neg\beta(y))\} \subseteq T \cup (K \setminus T) \cup \{(\exists y(\varphi(y) \wedge \neg\beta(y))\}$ is satisfiable as needed.

Thus, using the compactness theorem, $T \cup p(c) \cup \{\exists y(\varphi(y) \wedge \neg\beta(y))\}$ is satisfiable for any $\varphi(y)$ and some $\beta(y) \in q(y)$. Thus $q(y)$ is not isolated for the theory $T \cup p(c)$, so, using the omitting type theorem, a model of $T \cup p(c)$ exists such that $q(y)$ is omitted. Clearly every model of $T \cup p(c)$ realizes $p(x)$, namely by the element c. A model of $T \cup p(c)$ omitting $q(x)$ also satisfies T, so such a model satisfies T realizes $p(x)$ and omits $q(x)$.