MODEL THEORY: HOMEWORK 4

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2: Show that if $\mathbb{A}$ is atomic and $U \subseteq A$ is a finite subset of the set underlying $\mathbb{A}$, then $\mathbb{A}_U$ is atomic.

Solution:
Let $U \subseteq A$ be a finite subset of the set underlying the $\mathcal{L}$ structure $\mathbb{A}$. Let $\bar{u}$ be an enumeration of U.

Since $\mathbb{A}$ is atomic, every type is isolated. In particular, $\mathrm{tp}^{\mathbb{A}}(\bar{a}\bar{u})$ is isolated for any $\bar{a} \in A$. Thus there is a formula $\varphi(\bar{x}, \bar{y})$ that isolates $\mathrm{tp}^{\mathbb{A}}(\bar{a}\bar{u})$, i.e. $\varphi(\bar{x}, \bar{y}) \in \mathrm{tp}^{\mathbb{A}}(\bar{a}\bar{u})$ is such that for every $\psi(\bar{x}, \bar{y}) \in \mathrm{tp}^{\mathbb{A}}(\bar{a}\bar{u})$,

$$\mathbb{A} \models \forall\bar{x}\forall\bar{y}(\varphi(\bar{x}, \bar{y}) \to \psi(\bar{x}, \bar{y})).$$

We will show that $\varphi(\bar{x}, \bar{u}) \in \mathcal{L}_U$ isolates $\mathrm{tp}^{\mathbb{A}_U}(\bar{a})$.

Note that $\psi(\bar{x}, \bar{y}) \in \mathrm{tp}^{\mathbb{A}}(\bar{a}\bar{u})$ if and only if $\psi(\bar{x}, \bar{u}) \in \mathrm{tp}^{\mathbb{A}_U}(\bar{a})$. Thus $\varphi(\bar{x}, \bar{y}) \in \mathrm{tp}^{\mathbb{A}}(\bar{a}\bar{u})$ implies $\varphi(\bar{x}, \bar{u}) \in \mathrm{tp}^{\mathbb{A}_U}(\bar{a})$. Let $\theta(\bar{x}) \in \mathrm{tp}^{\mathbb{A}_U}(\bar{a})$. Then $\theta(\bar{x}) = \psi(\bar{x}, \bar{u})$ for some $\psi(\bar{x}, \bar{u}) \in \mathrm{tp}^{\mathbb{A}}(\bar{a}\bar{u})$. Therefore,

$$\mathbb{A} \models \forall\bar{x}\forall\bar{y}(\varphi(\bar{x}, \bar{y}) \to \psi(\bar{x}, \bar{y})).$$

Hence

$$\mathbb{A} \models \forall\bar{x}(\varphi(\bar{x}, \bar{y}) \to \psi(\bar{x}, \bar{y}))[\bar{u}].$$

So we have that

$$\mathbb{A}_U \models \forall\bar{x}(\varphi(\bar{x}, \bar{u}) \to \theta(\bar{x})).$$

Since this holds for an arbitrary $\theta(\bar{x}) \in \mathrm{tp}^{\mathbb{A}_U}(\bar{a})$, we have that $\varphi(\bar{x}, \bar{u})$ isolates $\mathrm{tp}^{\mathbb{A}_U}(\bar{a})$. Thus $\mathbb{A}_U$ is atomic.

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