1. The theory T of countably many independent unary relations is the theory in the language with relation symbols $R_n(x)$, $n < \omega$, which contains all sentences of the form

$$\exists x(R_{i_1}(x) \wedge \cdots \wedge R_{i_m}(x) \wedge \neg R_{j_1}(x) \wedge \cdots \wedge \neg R_{j_n}(x))$$

whenever $i_1, \ldots, i_m, j_1, \ldots, j_n$ are distinct.

(a) Show that T has quantifier elimination and is complete.

(b) Derive from (a) that any n-type is generated by formulas of the form $\pm$atomic, i.e., those of the form $x_i = x_j$, $x_i \neq x_j$, $R_i(x_j)$, and $\neg R_i(x_j)$.

(c) Explain why $S_n(T)$ has no isolated points, hence is homeomorphic to the Cantor set.

Part (a). Our first goal to prove that T has q.e. For this, it suffices to show that if $\varphi(x_1 \ldots, x_n)$ has the form $\exists y \theta(x_1, \ldots, x_n, y)$ for θ of the form $\bigwedge \pm$atomic, then $T \models \forall \bar{x}\,(\varphi(\bar{x}) \leftrightarrow \alpha(\bar{x}))$ for some quantifier-free formula α. For this we need to examine the structure of formulas of the form $\bigwedge \pm$atomic.

First some terminology: if $\mathbb{A} \models T$ and $a \in A$, then the isomorphism type of the substructure $\langle a \rangle$ is determined by specifying whether $a \in R_n^{\mathbb{A}}$ or $a \notin R_n^{\mathbb{A}}$ for each $n \in \omega$. This may be recorded as a binary number $.d_0 d_1 d_2 \ldots$, where $d_n = 0$ if $a \notin R_n^{\mathbb{A}}$ and $d_n = 1$ if $a \in R_n^{\mathbb{A}}$. Call this number the *binary expansion* of a and denote it $[a]$. Next, a formula in the variable x of the form $\bigwedge \pm$atomic which does not include $\pm(x = x)$ among its conjuncts, nor both of $R_n(x)$ and $\neg R_n(x)$ for any $n \in \omega$, must be a *finite partial description of the binary expansion of* x. Next, any formula of the form $\bigwedge \pm$atomic whose only conjuncts are of the form $\pm(x_i = x_j)$, which is consistent with T, must be a partial description of an equivalence relation on the variables involved. Finally, any formula of the form $\bigwedge \pm$atomic, which is not inconsistent with T, can be grouped as

$$\beta_1(x_1) \wedge \cdots \wedge \beta_n(x_n) \wedge \varepsilon$$

where each $\beta_i(x_i)$ is a partial description (possibly the empty description) of the binary expansion of x_i and ε is a partial description of an equivalence relation on $\{x_1, \ldots, x_n\}$.

Now we explain how to eliminate the quantifier from $\exists y \theta(\bar{x}, y)$ for θ of the form $\bigwedge \pm$atomic.

(1) If θ is inconsistent with T (no tuple of any model can satisfy θ), then replace $\exists y\theta(\bar{x}, y)$ with $x_1 \neq x_1$.

(2) If θ is consistent with T, and has the form

$$\beta_1(x_1) \wedge \cdots \wedge \beta_n(x_n) \wedge \beta(y) \wedge \varepsilon,$$

where ε contains a positive conjunct of the form $(y = x_i)$ for some i, then replace all instances of y in θ with x_i and note that $\exists y\theta(\bar{x}, y)$ is equivalent modulo T to the q.f. formula $\theta(\bar{x}, x_i)$.

(3) If θ is consistent with T, and has the form

$$\beta_1(x_1) \wedge \cdots \wedge \beta_n(x_n) \wedge \beta(y) \wedge \varepsilon,$$

where ε contains no positive conjuncts $(y = x_i)$ where y appears, then create $\alpha(\bar{x})$ from θ by eliminating all conjuncts from θ which involve y.

To see why Case (3) creates an equivalent q.f. formula, note that if θ has the form

$$\beta_1(x_1) \wedge \cdots \wedge \beta_n(x_n) \wedge \beta(y) \wedge \varepsilon,$$

where ε contains no positive conjuncts of the form $(y = x_i)$, then it expresses finite partial information about the binary expansions of each of $x_1, \ldots, x_n, y$, and partial information about the equalities that hold among them, but the only equality information involving y involves **in**equalities: clauses of the form $\neg(y = x_i)$. So it suffices to observe that in any model $\mathbb{A}$ of T, and any tuple $(a_1, \ldots, a_n) \in A^n$ which has any specified finite partial description of the binary expansions of and equalities between the coordinate values, there is always an element b distinct from all a_i satisfying any specified finite partial description of its binary expansion.
As an illustration of the last sentence, suppose that $\theta(x_1, x_2, x_3, y)$ expresses that the finite partial binary expansions of the variables are $[x_1] = .01101$, $[x_2] = .01101$, $[x_3] = .00111$, and $[y] = .01101$, and the partial description of the equivalence relation on $\{x_1, x_2, x_3, y\}$ expresses that they are all distinct. Then the formula $\exists y\theta(x_1, x_2, x_3, y)$ is satisfied by the same triples from models of T as the formula that expresses that $[x_1] = .01101$, $[x_2] = .01101$, $[x_3] = .00111$ and that the x's are distinct. The reason for this is that T guarantees that, for any triple (a_1, a_2, a_3) of distinct elements from a model whose expansions begin $[a_1] = .01101$, $[a_2] = .01101$, $[a_3] = .00111$, there always exists a b in the model whose binary expansion begins $[b] = .01101$ but then differs from each of $[a_1], [a_2]$ and $[a_3]$ in some later digit. The

formula that expressing that $[x_1] = .01101$, $[x_2] = .01101$, $[x_3] = .00111$ and that the x's are distinct is q.f.
Now we argue that T is complete. During each step where we eliminated $\exists y$ from a formula consistent with T we deleted all occurrences of y. When we deleted $\exists y$ from an inconsistent formula we also deleted all occurrences of y, but added $x_1 \neq x_1$. Eliminating existential and universal quantifiers from a sentence will result only in formulas involving $\pm(x_1 = x_1)$, and any such formula is equivalent to $\top$ or $\perp$. Thus every sentence is equivalent modulo T to a valid one or to a contradiction.

Part (b): Because the theory has quantifier elimination, any formula φ in an n-type can be written in quantifier-free disjunctive normal form: $\varphi = \bigvee m_i$, with each $m_i = \bigwedge a_j$ where the a_j's are $\pm$atomic. A formula in disjunctive normal form is satisfied by an n-tuple iff one of its monomials is satisfied by the n-tuple. This monomial is satisfied iff each of its $\pm$atomic formulas are satisfied. Then the n-type will include these $\pm$atomic formulas, from which the φ would be generated since it is a weaker statement than those $\pm$atomic formulas.

Part (c): Pick any n-formula φ and note that it can only reference at most finitely many relations. Pick a relation $R(x_i)$ not mentioned in φ, where x_i is a variable mentioned in φ. By the axioms of T, there are n-tuples satisfying both $\varphi \wedge R(x_i)$ and $\varphi \wedge \neg R(x_i)$. So, every n-formula is contained in at least two incompatible n-types so that there are no isolated points of $S_n(T)$. Thus, $S_n(T)$ is its own perfect kernel. Since the language of T is countable, then $S_n(T)$ is second countable. Since $S_n(T)$ is a Stone space, the Cantor-Bendixson Theorem covered in class guarantees that $S_n(T)$ is isomorphic to the Cantor set.