MODEL THEORY: HOMEWORK 1

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4 Let T be the theory of $\mathbb{A} = \langle \omega; \cdot, + \rangle$. Show that T is compatible with 2^ω distinct 1-types over the empty set. Conclude that there are 2^ω countable models of T up to isomorphism. (Find these as elementary extensions of $\mathbb{A}$.)

Claim 1: T is compatible with 2^ω distinct 1-types over the empty set.
Proof:
Let Q be the set of all prime numbers, which is known to have cardinality ω. This implies that the power set of Q has cardinality 2^ω. For each subset $K \subseteq Q$, define the set of formulas in one variable Σ_K to be $\{\exists y((x = y + y + ... + y) \wedge (x \neq y))\ for\ y\ added\ to\ itself\ i\ times | i \in K\} \cup \{\neg \exists y((x = y + y + ... + y))\ for\ y\ added\ to\ itself\ i\ times | i \in Q \setminus K\}$.
Subclaim: Σ_K is a partial 1-type.
Proof:
It suffices to show that $\Sigma_K \cup T$ is finitely satisfiable. For every finite subset F of $\Sigma_K \cup T$, only finitely many prime i_j exist such that $\exists y((x = y + y + ... + y))$ for y added to itself i times for $i \in K$ is an element of Σ_K. The product z of all such prime numbers satisfy all such sentences while also satisfying $\neg \exists y((x = y + y + ... + y))$ for y added to itself i times for all $i \in Q \setminus K$. Recognize that every product of prime numbers is an element of $\mathbb{A}$, so satisfy T. Thus a $z \in \omega$ exists satisfying $\Sigma_K \cup T$ as needed for Σ_K to be a partial $1-$type for every $K \subseteq Q$.

Recognize that when K and J are distinct subsets of Q, $K \setminus (K \cap J) \neq \emptyset$ or $J \setminus K \neq \emptyset$.

For $K \setminus (K \cap J) \neq \emptyset$, an element a exists. For this element, recognize that $\exists y((x = y + y + ... + y) \wedge (x \neq y))$ for y added to itself a times is an element of Σ_K and $\neg(\exists y((x = y + y + ... + y) \wedge (x \neq y)))$ for y added to itself a times is an element of Σ_J. No element can satisfy both of these formulas in any extension of $\mathbb{A}$, so no set of formulas containing both these is contained in a complete type meaning it is not a partial type or a complete type. So $\Sigma_K \cup \Sigma_J$ is neither a partial nor complete type.

For $J \setminus K$ nonempty, an element a exists. For this element, recognize that $\exists y((x = y + y + ... + y) \wedge (x \neq y))$ for y added to itself a times is an element of Σ_J and $\neg(\exists y((x = y + y + ... + y) \wedge (x \neq y)))$ for y added to itself a times is an element of Σ_K. No element can satisfy both of these formulas in any extension of $\mathbb{A}$, so no set of formulas containing both these is contained in a complete type so is not a partial type or a complete type.

This means that $\Sigma_K \cup \Sigma_J$ is not a complete or partial type for any $K \neq J$, so the complete types containing Σ_K and Σ_J are distinct for any $K \neq J$. Meaning distinct complete types exist containing Σ_K for each $K \subseteq Q$, so at least as many complete types exist as partial types of the form Σ_K for infinite K. That is to say, at least 2^ω complete types exist.

Claim 2: There are 2^ω distinct countable models of T up to isomorphism:
Proof:
For any infinite $K \subseteq \omega$, an element realizing Σ_K must be divisible by exactly the elements of K, and no such element exists in ω, so for every infinite K, Σ_K is not realized in ω. Thus every complete type containing Σ_K is omitted by $\mathbb{A}$ when K is infinite.

For any $1 - type$ of $\mathbb{A}$, a countable elementary extension $\mathbb{N}$ of $\mathbb{A}$ exists realizing it. For $\mathbb{N}$ to be a countable extension of $\mathbb{A} = \langle \omega; +, \cdot \rangle$, the set underlying $\mathbb{N}$ must be countable, thus only countably many such types can be realized because a single element cannot realize two distinct complete types one containing Σ_K and the other containing Σ_L for distinct infinite K and L.

Assume λ many countable elementary extensions exist. Then the set of partial types Σ_K realized in extensions of $\mathbb{A}$ is a union of λ many countable sets. The maximum size of this union occurs when every set is disjoint, which has size $\lambda \cdot \omega = \max\{\lambda, \omega\}$, otherwise the size is less than or equal to $\max\{\lambda, \omega\}$. Thus at most $\max\{\lambda, \omega\}$ many 1-types can be realized. $\max\{\lambda, \omega\} < 2^\omega$ whenever $\lambda < \omega$ because ω is always strictly less than 2^ω, so λ cannot be less than 2^ω. Meaning the number of countable elementary extensions λ must be at least 2^ω. Every elementary extension of $\mathbb{A}$ models T, so at least 2^ω countable models of T exist.