

2. Consider any L' -formula $\phi(\bar{x}, \bar{c})$ where \bar{x} represents free variables and \bar{c} represents constants. Then there is a corresponding L -formula $\phi(\bar{x}, \bar{y})$ where the constants have been replaced by free variables \bar{y} . Because T has quantifier elimination, then there is a quantifier-free L -formula $\psi(\bar{x}, \bar{y})$ such that $T \models \forall \bar{x}, \bar{y} (\phi(\bar{x}, \bar{y}) \leftrightarrow \psi(\bar{x}, \bar{y}))$. Because $T' \supseteq T$, then $T' \models \forall \bar{x}, \bar{y} (\phi(\bar{x}, \bar{y}) \leftrightarrow \psi(\bar{x}, \bar{y}))$. Since this is true for all free variables \bar{y} , we can simply replace them with constants \bar{c} and get that $T' \models \forall \bar{x} (\phi(\bar{x}, \bar{c}) \leftrightarrow \psi(\bar{x}, \bar{c}))$. Since $\psi(\bar{x}, \bar{c})$ is quantifier free, then T' has quantifier elimination.

If a theory lacks a statement about the order of two constants c_1 and c_2 , then clearly the theory cannot be complete since neither $c_1 < c_2$ nor $\neg(c_1 < c_2)$ is proved by the theory. Now consider a theory T that completely determines the order of the constants. Because DLO without endpoints has quantifier elimination, then by the argument above, adding constants to DLO still yields a theory with quantifier elimination. And for a theory with quantifier elimination, any embedding of a substructure into a larger structure will be elementary. Let \mathcal{M} and \mathcal{N} be two structures modeling T and consider the substructures of each consisting of just those constants. Then because T completely determines the order of the constants, the substructures must be isomorphic. And these substructures are elementary equivalent to \mathcal{M} and \mathcal{N} , so the structures satisfy the same formulas. If there was a formula ϕ such that neither $T \models \phi$ nor $T \models \neg\phi$, then we could create two models of the theory that were not elementary equivalent. Thus, T must be complete.