Math 6000 Model Theory

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Homework 3

1. You might conjecture that the composition of elementary embeddings is elementary, or that the composition of nonelementary embeddings is nonelementary. This problem explores all relationships of this type. Namely it explores chick maps, in a triple of embeddings $(f, g, g \circ f)$ can be (or are forced to be) elementary.

For each triple $(x, y, z) \in \{$ elementary, not$\}^3$ find an example

$$\mathbb{A} \xrightarrow{f} \mathbb{B} \xrightarrow{g} \mathbb{C}$$

realizing the triple, if possible, or explain why there is no example.

(elementary, elementary, not): Assume f, g are both elementary embeddings. Thus for any formula ψ and $\overline{a} \in A$, we have

$$\begin{aligned} & \mathbb{A} \models \psi[\overline{a}] \\ \iff & \mathbb{B} \models \psi[f(\overline{a})] \\ \iff & \mathbb{C} \models \psi[g(f(\overline{a}))]. \end{aligned}$$

Hence $g \circ f$ is elementary. Therefore there is no example satisfying this tuple.

(not, elementary, elementary): Assume f is not an elementary embedding and g is elementary. Since f is not elementary, there exists some formula $\psi(\overline{x})$ and some $\overline{a} \in A$ such that $\mathbb{A} \models \psi[\overline{a}]$ but $\mathbb{B} \not\models \psi[f(\overline{a})]$. Then we have $\mathbb{B} \models \neg\psi[f(\overline{a})]$ so $\mathbb{C} \models \neg\psi[g(f(\overline{a}))]$. Hence $\mathbb{C} \not\models \psi[g(f(\overline{a}))]$. Therefore $g \circ f$ is not elementary.

(not, not, elementary): Let $\mathcal{L} =<$ and take $\mathbb{A} = (0, \infty), \mathbb{B} = [0, \infty), \mathbb{C} = \mathbb{R}$. Also let $\mathbb{A} \xrightarrow{f} \mathbb{B}$ and $\mathbb{B} \xrightarrow{g} \mathbb{C}$ where f, g are both inclusion maps. They are both not elementary embeddings since $\mathbb{B}$ has the smallest element while $\mathbb{A}, \mathbb{C}$ do not. $f \circ g$ is the inclusion map from $\mathbb{A}$ to $\mathbb{C}$. In Theorem 3.1.4

take $M, A = \mathbb{A}$ and take $N = \mathbb{C}$, this is possible since $\mathbb{A}, \mathbb{C}$ are both models of DLO with no min nor max. Theorem 3.1.4(ii) gives us that $f \circ g$ is an elementary embedding.

(elementary, not, elementary):

Let $\mathbb{A} = \langle \mathbb{N}; < \rangle$ and $\mathbb{B} = \mathbb{C}$ be a proper elementary extension of $\mathbb{A}$. First note that the structure $\langle \mathbb{N}; < \rangle$ has the first-order expressible properties that (1) it is a strict linear order, (2) every element of $\mathbb{N}$ has a unique successor, (3) every element that is not the least element has a unique predecessor, and (4) for each n there is a formula $\varphi_n(x)$ which asserts that the elements $\leq x$ form an n-element chain.

Since $\mathbb{B}$ is a proper elementary extension of $\mathbb{A} = \mathbb{N}$, $\mathbb{B}$ has all of these properties. Each n-element initial segment of $\mathbb{B}$ must agree with that of $\mathbb{A}$, so $\mathbb{A}$ must be an initial segment of $\mathbb{B}$. Moreover if $x \in B - A$ the collection of iterated predecessors and successors of x form a copy of $\mathbb{Z}$ that lies entirely above all elements of $\mathbb{A}$. Thus $\mathbb{B}$ (and hence $\mathbb{C}$) has a copy of $\mathbb{Z}$ above $\mathbb{A}$. Let $b_i \in B$ denote the elements in the copy of $\mathbb{Z}$ such that the the successor of b_i with respect to the order is b_{i+1}.

Define $f : \mathbb{A} \to \mathbb{B}$ to be the inclusion function and and $g : \mathbb{B} \to \mathbb{C}$ to be $g(b_i) = 2b_i$ and $g(n) = n$ for all $n \neq b_i$ for some i. Clearly, f is an elementary embedding. Let $\psi[x_1, x_2]$ be the formula that says there does not exist an element between x_1 and x_2, i.e. the formula $\neg\exists y((x_1 < y) \vee (y < x_2))$. The map g is not elementary since $\mathbb{B} \models \psi[b_0, b_1]$ but $\mathbb{C} \not\models \psi[g(b_0), g(b_1)]$. However the composition map $g \circ f$ is the inclusion function, and thus is an elementary embedding.

For the following cases we will let $\mathbb{A}, \mathbb{B}, \mathbb{C} = \langle \mathbb{N}, < \rangle$. Define the formula $\varphi(x)$ to be the formula that says x is a least element, i.e.

$$\varphi(x) : \forall y((x < y) \vee (x = y)).$$

(elementary, elementary, elementary): Let f, g be the identity maps. Thus f, g are elementary embeddings. Then clearly $g \circ f$ is an elementary embeddings.

(not, elementary, not): Let f be defined by $f(n) = n+1$ and g be the identity map. Then f is not elementary since $\mathbb{A} \models \varphi[0]$ but $\mathbb{B} \not\models \varphi[1]$. Thus composition function $g \circ f$ given by $(g \circ f)(n) = n+1$ is also not elementary.

(elementary, not, not): Similar argument to the (not, elementary, not) case holds now letting g to be defined by $g(n) = n+1$ and f be the identity map.

(not, not, not): Let f be defined by $f(n) = n+1$ and $g = f$. We have shown that f and g are not elementary. The composition function $g \circ f$ given by $(g \circ f)(n) = n + 2$ is also not elementary since $\mathbb{A} \models \varphi[0]$ but $\mathbb{C} \not\models \varphi[2]$.