MODEL THEORY SPRING 2016 ASSIGNMENT 2 PROBLEM 7

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Let $\mathscr{U}$ be an ultrafilter on an index set I, and let $\{\mathbf{A_i}\}_{i\in I}$ be a set of $\mathscr{L}$-structures for a language $\mathscr{L}$. We denote by q the canonical projection map $\prod_{i\in I} A_i \to \prod_{\mathscr{U}} A_i$, we write $A \hookrightarrow B$ for injective maps of sets and we write $\mathbf{A} \hookrightarrow \mathbf{B}$ for an embedding of $\mathscr{L}$-structures.

We note that for an $\mathscr{L}-$structure $\mathbf{A}$, the set

$$\mathsf{Aut}\left(\mathbf{A}\right) = \left\{ f : \mathbf{A} \hookrightarrow \mathbf{A} \middle| \, f : A \hookrightarrow A \text{ is a surjection}\right\}$$

is a structure in the language of groups $\mathscr{L}_{\mathsf{Grp}} = \left\{R.\left(\_,\_,\_\right), R_{\mathrm{id}}\left(\_\right)\right\}$ in the obvious way.

Lemma 0.1. *There is a natural embedding of the $\mathscr{L}_{\mathsf{Grp}}$-structures*

$$\alpha : \prod_{\mathscr{U}} \mathsf{Aut}\left(\mathbf{A}_i\right) \hookrightarrow \mathsf{Aut}\left(\prod_{\mathscr{U}} \mathbf{A}_i\right).$$

Proof. We may assume that $\mathscr{L}$ is a purely relational language. Thus any $\mathscr{L}$-structure $\mathbf{A}_i$ may be given as

$$\mathbf{A}_i = \langle A_i; \mathscr{R}_i\rangle$$

where $\mathscr{R}_i$ is a set of subsets of A_i^k for varying arity k. We may then write that

$$\mathsf{Aut}\left(\mathbf{A}_i\right) = \left\{ f : A_i \xrightarrow{\sim} A_i \middle| \forall \overline{a} \in A_i \forall R \in \mathscr{R}_i \left(R\left(\overline{a}\right) = R\left(f\left(\overline{a}\right)\right)\right)\right\}.$$

Thus a member of the ultra-product $\prod_{\mathscr{U}} \mathsf{Aut}\left(\mathbf{A}_i\right)$ is an equivalence class $\left[\left(f_i\right)_{i\in I}\right]_{\mathscr{U}}$ where

$$\left(f_i\right)_{i\in I} \sim_{\mathscr{U}} \left(g_i\right)_{i\in I}$$

if and only if

$$\left\{ i \in I \middle| \, f_i = g_i\right\} \in \mathscr{U}.$$

Observe then that for any

$$\left(f_i\right)_{i\in I} \in \left[\left(f_i\right)_{i\in I}\right]_{\mathscr{U}} \in \prod_{\mathscr{U}} \mathsf{Aut}\left(\mathbf{A}_i\right)$$

the composition $q \circ f_i \circ \mathsf{Pr_i}$ preserves $\mathscr{U}$-classes. Indeed suppose $\left(a_i\right) \sim_{\mathscr{U}} \left(b_i\right)$, then as

$$\left(a_i\right) \xmapsto{f_i \circ \mathsf{Pr_i}} \left(f_i\left(a_i\right)\right) \xmapsto{q} \left[f_i\left(a_i\right)\right],$$

likewise $\left(b_i\right) \mapsto \left[f_i\left(b_i\right)\right]$, and what's more $\left\{ i \in I \middle| \, a_i = b_i\right\} \in \mathscr{U}$ we find that

$$\left\{ i \in I \middle| \, f_i\left(a_i\right) = f_i\left(b_i\right)\right\} \in \mathscr{U}$$

so $\left[f_i\left(a_i\right)\right] = \left[f_i\left(b_i\right)\right]$. Thus we've the unique dashed completion f in the diagram below

$$\begin{tikzcd}
\prod_{i\in I} A_i \arrow[r, "\prod f_i \circ \mathsf{Pr_i}"] \arrow[d, "q"'] & \prod_{i\in I} A_i \arrow[r, "q"] & \prod_{\mathscr{U}} A_i. \\
\prod_{\mathscr{U}} A_i \arrow[urr, dashed, "\exists! f"'] & &
\end{tikzcd}$$

If $(f_i)_{i\in I} \sim_{\mathscr{U}} (g_i)_{i\in I}$ then $\prod_{i\in I} f_i \circ \mathsf{Pr}_i$ agrees with $\prod_{i\in I} g_i \circ \mathsf{Pr}_i$ in almost all coordinates and their respective compositions with the canonical projection q induce the same map $\prod_{\mathscr{U}} A_i \to \prod_{\mathscr{U}} A_i$.

These induced maps define automorphisms of the ultraproduct $\prod_{\mathscr{U}} \mathbf{A}_i$. Suppose $f = \alpha([(f_i)])$, then f is bijective: indeed f is injective as the f_i are automorphisms so

$$\begin{aligned} f([(a_i)]) = f([(b_i)]) &\leftrightarrow \{i \in I \,|\, f_i(a_i) = f_i(b_i)\} \in \mathscr{U} \\ &\leftrightarrow \{i \in I \,|\, a_i = b_i\} \in \mathscr{U} \\ &\leftrightarrow [(a_i)] = [(b_i)]; \end{aligned}$$

and surjective as for any $[(a_i)]$,

$$\begin{aligned} f\left(\left[\left(f_i^{-1}(a_i)\right)\right]\right) &= \left[\left(f_i\left(f^{-1}(a_i)\right)\right)\right] \\ &= [(a_i)]. \end{aligned}$$

This f is moreover an embedding as for any $R \in \mathscr{R}$ of arity k,

$$\begin{aligned} R\left(\left([(a_i^j)]\right)_{j=1}^k\right) &\leftrightarrow \left\{i \in I \,\middle|\, R\left(\left(a_i^j\right)_{j=1}^k\right)\right\} \in \mathscr{U} \\ &\leftrightarrow \left\{i \in I \,\middle|\, R\left(\left(f_i\left(a_i^j\right)\right)_{j=1}^k\right)\right\} \in \mathscr{U} \\ &\leftrightarrow R\left(\left(\left[f_i\left(a_i^j\right)\right]\right)_{j=1}^k\right) \end{aligned}$$

by virtue of the f_i being automorphisms. We may then define the map α by

$$\begin{aligned} \textstyle\prod_{\mathscr{U}} \mathsf{Aut}(\mathbf{A}_i) &\xrightarrow{\alpha} \textstyle\prod_{\mathscr{U}} \mathsf{Aut}(\mathbf{A}_i) \\ \left[(f_i)_{i\in I}\right]_{\mathscr{U}} &\longmapsto f. \end{aligned}$$

The map α is injective. Indeed if $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ induce in this manner the same map then the compositions

$$\textstyle\prod_{i\in I} A_i \xrightarrow{\prod f_i \circ \mathsf{Pr}_i} \prod_{i\in I} A_i \xrightarrow{q} \prod_{\mathscr{U}} A_i$$

and

$$\textstyle\prod_{i\in I} A_i \xrightarrow{\prod g_i \circ \mathsf{Pr}_i} \prod_{i\in I} A_i \xrightarrow{q} \prod_{\mathscr{U}} A_i$$

agree, meaning that they agree in almost all indices prior to composition with q; i.e.

$$\{i \in I |\, f_i \circ \mathsf{Pr}_i = g_i \circ \mathsf{Pr}_i\} \in \mathscr{U}$$

which puts

$$\{i \in I |\, f_i = g_i\} \in \mathscr{U}$$

as those are the same set since the maps Pr_i are epimorphisms.

The interpretation of composition, seen as an interpretation of R. in the language of groups in $\mathsf{Aut}(\prod_{\mathscr{U}} \mathbf{A}_i)$ is clear, likewise the interpretation of R. in $\prod_{\mathscr{U}} \mathsf{Aut}(\mathbf{A}_i)$ is given;

$$R.\left((x_i), (y_i), (z_i)\right) \leftrightarrow \{i \in I |\, x_i \circ y_i = z_i\} \in \mathscr{U}.$$

We observe moreover that for fixed k

$$\begin{tikzcd} \prod_{i\in I} A_i \arrow[r, "\prod x_i \circ \mathsf{Pr_i}"] \arrow[d, "\mathsf{Pr}_k"'] & \prod_{i\in I} A_i \arrow[d, "\mathsf{Pr_k}"] & \\ A_k \arrow[r, "x_k"'] & A_k \arrow[r, "y_k"] & A_k \end{tikzcd}$$

commutes so the inducement α preserves composition, and if $[(x_i)] = [(\mathrm{id}_{A_i})]$ then

$$\alpha([(x_i)]) = \alpha([(\mathrm{id}_{A_i})])$$

which is induced by

$$\prod_{i\in I} A_i \xrightarrow{\prod \mathrm{id}_{A_i} \circ \mathsf{Pr_i}} \prod_{i\in I} A_i$$

which is but the identity map on the product as it is so in every index so α preserves the the identity; to wit α is functorial.

That functoriality provides that α is an embedding of $\mathscr{L}_{\mathsf{Grp}}$-structures. Indeed we've both that

$$\begin{aligned} R_{\cdot}([(x_i)], [(y_i)], [(z_i)]) &\leftrightarrow \{i \in I \,|\, x_i \circ y_i = z_i\} \in \mathscr{U} \\ &\leftrightarrow \alpha((x_i \circ y_i)) = \alpha((z_i)) \\ &\leftrightarrow \alpha((x_i)) \circ \alpha((y_i)) = \alpha((z_i)) \\ &\leftrightarrow R_{\cdot}(\alpha([(x_i)]), \alpha([(y_i)]), \alpha([(z_i)])), \end{aligned}$$

and

$$\begin{aligned} R_{\mathrm{id}}([(x_i)]) &\leftrightarrow \{i \in I \,|x_i = \mathrm{id}_{A_i}\} \in \mathscr{U} \\ &\leftrightarrow \alpha([(x_i)]) = \alpha([(\mathrm{id}_{A_i})]) \\ &\leftrightarrow \alpha([(x_i)]) = \mathrm{id}_{\prod_{\mathscr{U}} A_i} \\ &\leftrightarrow R_{\mathrm{id}}(\alpha([(x_i)])). \end{aligned}$$

□

This is embedding is not in general an isomorphism; consider the following example. Let $\mathscr{L}$ be the language

$$\mathscr{L} = \{R_=\} \cup \{f_i\}_{i\in\omega}$$

where f_i is a unary function symbol for each i, and for each $i \in \omega$ we let

$$A_i = \{0, 1\},$$

and let $\mathscr{U}$ be a non-principal ultra-filter on ω. We define ω-many $\mathscr{L}$-structures $\mathbf{A}_i$ with domains A_i, interpreting the relation of equality in the obvious way, and the interpretation of the function symbols f_i in $\mathbf{A}_j$ given in table.

We note then that for each $i \in \omega$, the group $\mathsf{Aut}(\mathbf{A}_i)$ is trivial, as any automorphism φ must commute with all function symbols and in particular with f_i, but then

$$\begin{aligned} \varphi(0) &= \varphi(f_i(1)) \\ &= f_i(\varphi(1)) \\ &= 0 \end{aligned}$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
f_3	$\mathrm{id}_{\{0,1\}}$	$\mathrm{id}_{\{0,1\}}$	0	$\cdots$
f_2	$\mathrm{id}_{\{0,1\}}$	0	$\mathrm{id}_{\{0,1\}}$	$\cdots$
f_1	0	$\mathrm{id}_{\{0,1\}}$	$\mathrm{id}_{\{0,1\}}$	$\cdots$
	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\cdots$

TABLE 1. Interpretations of function symbols f_i

which forces $\varphi = \mathrm{id}_{\{0,1\}}$. Since $\mathsf{Aut}\,(\mathbf{A}_i)$ is trivial for all $i \in \omega$, $\prod_{\mathscr{U}} \mathsf{Aut}\,(\mathbf{A}_i)$ is trivial and to prove the properness of the embedding $\prod_{\mathscr{U}} \mathsf{Aut}\,(\mathbf{A}_i) \longrightarrow \mathsf{Aut}\,(\prod_{\mathscr{U}} \mathbf{A}_i)$ it then suffices to show that $\mathsf{Aut}\,(\prod_{\mathscr{U}} \mathbf{A}_i)$ admits a single non-trivial element. We construct this automorphism explicitly.

First, observe that the function symbols f_j interpreted in $\prod_{\mathscr{U}} \mathbf{A}_i$ are the identity function. Let φ be the induced map in the diagram

$$\begin{tikzcd}
\prod_{i\in I} A_i \arrow[r, "\prod\left\{\begin{array}{c} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}\right\}\circ \mathsf{Pr}_i"] \arrow[d, "q"'] & \prod_{i\in I} A_i \arrow[r, "q"] & \prod_{\mathscr{U}} A_i. \\
\prod_{\mathscr{U}} A_i \arrow[urr, dashed] & &
\end{tikzcd}$$

The map φ is indeed a bijective map of sets, but it is more, it is an embedding of $\mathscr{L}$-structures: the preservation and reflection of $R_=$ is clear and our observation that $f_j^{\prod_{\mathscr{U}} \mathbf{A}_i} = \mathrm{id}_{\prod_{\mathscr{U}} \mathbf{A}_i}$ makes immediate the commutation of those functions with any function, and in particular with φ. To wit φ is a non-trivial automorphism and the embedding $\prod_{\mathscr{U}} \mathsf{Aut}\,(\mathbf{A}_i) \longrightarrow \mathsf{Aut}\,(\prod_{\mathscr{U}} \mathbf{A}_i)$ is not in general an isomorphism.